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Slow Motion and Metastability for a Nonlocal Evolution Equation

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In this paper we consider a nonlocal evolution equation in one dimension, which describes the dynamics of a ferromagnetic system in the mean field approximation. In the presence of a small magnetic field, it admits two stationary and homogeneous solutions, representing the stable and metastable phases of the physical system. We prove the existence of an invariant, one dimensional manifold connecting the stable and metastable phases. This is the unstable manifold of a distinguished, spatially nonhomogeneous, stationary solution, called the critical droplet.(4, 10) We show that the points on the manifold are droplets longer or shorter than the critical one, and that their motion is very slow in agreement with the theory of metastable patterns. We also obtain a new proof of the existence of the critical droplet, which is supplied with a local uniqueness result.

KEY WORDS: Critical droplet; phase transition; unstable manifold.

1. INTRODUCTION

In this paper we study the metastable behavior in one dimension of the following evolution equation for the scalar field $m = m(t, x)$:

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta[J * m + h]\}, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{1.1}$$

where $\beta > 1$, $h > 0$, and J is a nonnegative, even function, with support in the unit interval, normalized to have integral 1, i.e., $\int dx\, J(x) = 1$, and

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$J * m$ denotes the convolution between J and m. Equation (1.1) has been derived in ref. 11, as a scaling limit for the empirical magnetization, from an Ising spin system with Glauber dynamics and Kac potentials.[18–20] Therefore m represents the local magnetization density, J is related to the (long range) ferromagnetic coupling of the spin–spin interaction, β is the inverse temperature and h an external magnetic field. The appearance of metastable states is a common feature of all the mean field theories of phase transition. This fact is reflected in (1.1): given $\beta > 1$, there is an $h_\beta > 0$ such that for $h \in [0, h_\beta]$ there are three and only three constant stationary solutions, denoted by:

$$m^-_{\beta, h} < m^0_{\beta, h} \leqslant 0 < m^+_{\beta, h}.$$

For $h > 0$, $|m^-_{\beta, h}| < m^+_{\beta, h}$, and $m^0_{\beta, h} < 0$; for $h = 0$, $m^0_{\beta, h} = 0$, and $m^+_{\beta, h} = -m_{\beta, h} \doteq m_\beta$. The two phases $\pm m_\beta$ are thermodynamically stable at $h = 0$, while $m^0_{\beta, h} = 0$ is unstable. For $h > 0$, $m^+_{\beta, h}$ is the only stable phase, $m^0_{\beta, h}$ is still unstable, while $m^-_{\beta, h}$ becomes metastable.

We prove the existence of an invariant manifold connecting the stable and metastable phases. This manifold is the union of two heteroclinic orbits, connecting the above phases with a particular, nonhomogeneous stationary solution to (1.1), which we call the critical droplet for the following reason.

Our ultimate purpose is to give a complete description of the tunneling from the metastable to the stable phase for the underlying stochastic spin dynamics. According to general heuristic arguments, the transition occurs through the *nucleation* of a sufficiently large droplet of the stable phase, which will start to grow undergoing an irreversible process leading to the stable phase everywhere. On the contrary, small droplets will have a tendency to shrink. According to the pathwise approach to metastability in the case of reversible dynamics,[7, 16, 21] our results will play a central role in determining the typical path of the tunneling transition, which will be the subject of future works.

The critical droplet is a spatially nonhomogeneous, symmetric function q which is a stationary solution to Eq. (1.1) close to the metastable state at infinity; namely q solves:

$$q(x) = \tanh\{\beta[(J * q)(x) + h]\}, \quad x \in \mathbb{R}, \qquad \lim_{|x| \to \infty} q(x) = m^-_{\beta, h}. \tag{1.2}$$

The existence of homoclinic type solutions like q has been firstly proved in ref. 4 for a general class of bistable integral equations (and general kernels J), which also give the stationary solutions of a different class of nonlocal evolution equations introduced in ref. 1. For our particular equation the existence has also been proved in ref. 10 for $\beta > 1$ and h small enough. Both the results in ref. 10 and ours relay on the spectral estimates given in ref. 9 that are proved only for h small and for positive, monotonic, compact support J. However this is enough for our purposes since the range of the spin–spin interaction is finite and the metastable behavior is caught only for h small.

In ref. 10 it has been proved that when h is very small the region where q is close to the stable phase is of order $|\log h|$, while the length of the transition layers from this region to the one where the metastable phase dominates is of order 1. The layers may be approximately described as standing waves of Eq. (1.1) with $h = 0$. These are translations and/or reflections of the *instanton* $\bar{m}(x)$, a strictly increasing and antisymmetric function solving:

$$\bar{m}(x) = \tanh\{\beta(J * \bar{m})(x)\}, \quad x \in \mathbb{R}, \qquad \lim_{x \to \pm\infty} \bar{m}(x) = \pm m_\beta. \tag{1.3}$$

All the translations of q are stationary solutions, and it is not known whether these are the only (spatially nonhomogeneous) solutions to (1.2). In ref. 10 the existence of the bump is proved by applying the Newton method to find the zeros of the map defined (on the space of continuous and symmetric functions) by the r.h.s. of (1.1). Consequence of this approach is the lack of any uniqueness result. We partially fill up this gap here by giving a different proof which is also supplied with a local uniqueness result. We also give a detailed description of the spatial structure of the bump, by showing it is a strictly decreasing function for $x > 0$, converging exponentially fast to the metastable phase as $|x| \to +\infty$.

The existence of an invariant, one dimensional, unstable manifold $\mathscr{W}$ through q, follows in a standard way (see, for instance, ref. 17) from the existence, proved in ref. 9, of an isolated, simple, positive eigenvalue of the operator obtained by linearizing (1.2) around the critical droplet q. Nevertheless we give in Section 6 an explicit construction needed to establish the other results. This manifold consists of the union of two branches $\mathscr{W}_\pm$. The points on $\mathscr{W}_-$ (resp. $\mathscr{W}_+$) are symmetric functions, nonincreasing for $x > 0$, strictly smaller (resp. larger) than q. We thus refer to the points on $\mathscr{W}_-$ (resp. $\mathscr{W}_+$) as the sub-critical (resp. super-critical) droplets. In the branch $\mathscr{W}_-$ where the length of the droplets is shorter than that of q, the evolution shrinks it further, while it grows if it is larger. This shrinking (resp. growing) process goes on indefinitely and we actually prove the branch $\mathscr{W}_-$ (resp. $\mathscr{W}_+$) connects the bump with the metastable (resp. stable) phase. We prove this last result by using the comparison theorem which holds for

Eq. (1.1). We construct suitable upper solutions for the sub-critical droplet and lower solutions for the super-critical one that converge to the metastable, respectively stable, phase as t diverges. This method does not allow to catch the actual speed of convergence to the metastable and stable phases, which will be the subject of a future work.

However we have a detailed description of large part of the relaxation process. The situation is in fact similar to that of the invariant manifolds for metastable patterns in solutions of the singularly perturbed Ginzburg–Landau equation.[5, 6, 15] We show that, for h very small, a large part, $\bar{\mathscr{W}}$, of the unstable manifold $\mathscr{W}$ consists of droplets with two well-separated transition layers, whose patterns are described approximately by the standing waves. The dynamics on $\bar{\mathscr{W}}$ is then reduced to the motion of the layer locations, $\pm\xi$, which can be described, to high accuracy, by the ordinary differential equation:

$$\dot{\xi} = -\mu K e^{-2\alpha\xi} + 2m_\beta \mu h, \tag{1.4}$$

where the positive coefficients K, α, and μ depend on J and β. Thus the velocity turns out to be the sum of two terms. The first one is an attractive term, due to the interaction between the layers, in agreement to the analogous effect proved for the singularly perturbed Ginzburg–Landau equation. The second one is a constant, positive drift, which is linear in the small magnetic field h. It coincides with the first order expansion of the velocity for the traveling wave solutions to (1.1), see Eq. (2.8) in the next section. Since the motion is very slow (for large ξ), following ref. 15 we refer to $\bar{\mathscr{W}}$ as a *slow motion manifold*. The velocity field has a zero at $(2\alpha)^{-1} \log[K(2m_\beta h)^{-1}]$ which gives, at the lowest order, (one half of) the length of the critical droplet.

Our approach is based on the geometric method used by Fusco and Hale,[15] and by Carr and Pego,[5, 6] for the singularly perturbed Ginzburg–Landau equation (see also the more recent paper by Eckmann and Rougemont[14]). By restricting our analysis to the space of symmetric functions we first construct an "approximately invariant" manifold $\mathscr{M}$ of quasi-stationary states. The motion near $\mathscr{M}$ is described in terms of coordinates along and transversal to $\mathscr{M}$ (the Fermi coordinates). This manifold contains the essential dynamics: due to a strong transverse stability, the solutions near $\mathscr{M}$ are squeezed into a *thin channel* which contains $\mathscr{M}$ and where the layer motion is described with high accuracy by (1.4). We next prove that there exists an unique stationary solution to Eq. (1.1) near $\mathscr{M}$, which is then identified with the critical droplet q and it turns out to be strictly contained in the thin channel. The slow motion manifold $\bar{\mathscr{W}}$ is thus defined by that part of $\mathscr{W}$ inside the channel.

In the next section we state the main definitions and results and we give an outline of the paper.

2. DEFINITIONS AND RESULTS

We first state the assumptions on the interaction J appearing in (1.1). The function $J \in C^3(\mathbb{R})$ is symmetric, nonnegative, and with integral equal to one. Moreover $\sup\{x \in \mathbb{R} : J(x) > 0\} = 1$ and $J'(x) < 0$ for $x \in (0, 1)$.

In the whole paper we consider the evolution defined by (1.1) as an equation in the space $L_\infty(\mathbb{R}; [-1, 1])$, which we rewrite as:

$$\frac{dm}{dt} = f(m), \qquad f(m) \doteq -m + \tanh\{\beta[J * m + h]\}. \tag{2.1}$$

We observe that the Cauchy problem has a unique solution in $L_\infty(\mathbb{R})$ and the set $L_\infty(\mathbb{R}; [-1, 1])$ is invariant for the dynamics. Analogously there exists a unique solution of the Cauchy problem in $C_0(\mathbb{R})$, the space of continuous and bounded functions on $\mathbb{R}$, and the set $C(\mathbb{R}, [-1, 1])$ is left invariant. We denote by $S_t(m)$ the flow solution with initial datum m, so that S_t defines a semi-group on $L_\infty(\mathbb{R}; [-1, 1])$ for which $C(\mathbb{R}, [-1, 1])$ is an invariant (closed) subspace. We finally notice that, since J is a symmetric function, the space $C^{\text{sym}}(\mathbb{R}, [-1, 1])$ of symmetric, continuous functions is an invariant set. In the sequel we will often need to study the dependence of $S_t(m)$ on the initial datum m. A first estimate is:

$$\|S_t(m+u) - S_t(m)\|_\infty \leqslant e^{k_1 t} \|u\|_\infty, \tag{2.2}$$

where $k_1 > 0$ is the Lipschitz coefficient of f. For a more refined bound, let L_m be the linear operator defined by

$$L_m u = -u + p_m J * u, \qquad p_m \doteq \frac{\beta}{\cosh^2\{\beta J * m\}}. \tag{2.3}$$

Then L_{m+h} is the derivative $Df|_m$ of $f(m)$ at m, so that:

$$\begin{aligned} f(m+u) - f(m) - L_{m+h} u \\ = \beta^2 (J * u)^2 \int_0^1 ds (1-s) \tanh''\{\beta[J * (m+su) + h]\}, \end{aligned} \tag{2.4}$$

from which it follows that there exists $k_2 > 0$ such that:

$$\|f(m+u) - f(m) - L_{m+h} u\|_\infty \leqslant k_2 \|u\|_\infty^2. \tag{2.5}$$

Stationary Solutions (Refs. 1, 3, 4, 8, 9, 10, 12, 13, 22). Given $\beta > 1$, there exists a unique (modulo translations) solution $\bar{m}(x)$ of (1.3), which we call the *instanton*. It is a C^∞, strictly increasing, antisymmetric function. Moreover, letting $\alpha > 0$ be such that:

$$\beta(1-m_\beta^2) \int dz\, J(z)\, e^{-\alpha z} = 1, \tag{2.6}$$

there are $a > 0$, $\alpha_0 > \alpha$, and $c > 0$ so that, for all $x \geqslant 0$,

$$|\bar{m}(x) - (m_\beta - a e^{-\alpha x})| + |\bar{m}'(x) - \alpha a e^{-\alpha x}| + |\bar{m}''(x) + \alpha^2 a e^{-\alpha x}| \leqslant c e^{-\alpha_0 x}, \tag{2.7}$$

where $\bar{m}'$ and $\bar{m}''$ are respectively the first and second derivatives of $\bar{m}$.

For small $h > 0$, the existence of traveling waves solutions to (1.1), connecting the stable and metastable phases $m^{\pm}_{\beta, h}$, can be obtained by using a perturbative argument around the instanton. These are solutions of the form $\tilde{m}_h(x - v(h)\, t)$, where $\tilde{m}_h(x)$ is a strictly increasing function of x with asymptotes $m^{\pm}_{\beta, h}$ at $\pm\infty$ and $\|\tilde{m}_h - \bar{m}\|_\infty = O(h)$. The velocity of the front $v(h)$ is a negative, strictly decreasing function of h, such that:

$$\lim_{h \to 0^+} \frac{v(h)}{h} = -2 m_\beta\, \mu, \qquad \mu \doteq \left[\int dy\, \frac{\bar{m}'(y)^2}{\beta(1-\bar{m}(y)^2)} \right]^{-1}. \tag{2.8}$$

On a physical background, μ is the linear transport coefficient which represents the *mobility* of an interface separating the stable phases, see, e.g., refs. 2 and 23 for details. We refer to ref. 8 and reference therein, for stability and other properties of fronts.

In ref. 10 it is proved the existence, for $h > 0$ small, of the critical droplet in a neighborhood of the symmetric function $\bar{m}_\xi(x) \doteq \bar{m}(\xi - |x|)$; the result is summarized in the following theorem.

Theorem 2.1. Given $\beta > 1$, there is $h_0 > 0$ and, for any $h \in (0, h_0]$, there is $q \in C^{\mathrm{sym}}(\mathbb{R}; [-1, 1])$ which solves (1.2). Moreover there is $\xi^* = \xi^*(h)$ such that:

$$\lim_{h \downarrow 0} \|q - \bar{m}_{\xi^*}\|_\infty = 0 \qquad \text{and} \qquad \lim_{h \downarrow 0} e^{2\alpha \xi^*(h)} h = (2 m_\beta)^{-1} K, \tag{2.9}$$

where

$$K \doteq \frac{a}{\beta(1-m_\beta^2)} \int dx\, \frac{e^{\alpha x} \bar{m}'(x)(m_\beta^2 - \bar{m}^2(x))}{1 - \bar{m}^2(x)}, \tag{2.10}$$

with α defined by (2.6) and a as in (2.7).

The Quasi-Invariant Manifold. We are interested in the behavior of solutions which consist of two well separated layers, and from (2.9) we expect that the interesting part of the relaxation process is caught by restricting our analysis to patterns with layers at a distance no too larger than $(2\alpha)^{-1} |\log h|$. In fact layers far apart from each other interact too weakly and the effect of the magnetic field predominates (one could prove they simply go away with an almost constant velocity, each one eventually approaching a traveling wave). With this in mind, for any $\ell > 1$, $h < e^{-2\alpha\ell}$, and $\kappa > 0$, we introduce the interval:

$$\Gamma_{\ell,\kappa} \doteq \{\xi : \ell < \xi < (2\alpha)^{-1} |\log h| + \kappa\}. \tag{2.11}$$

Observe that:

$$h < e^{2\alpha\kappa} e^{-2\alpha\xi} \qquad \forall \xi \in \Gamma_{\ell,\kappa}. \tag{2.12}$$

Theorem 2.2. Given $\beta > 1$, there is $\ell_0 > 1$ such that, for any $h < e^{-2\alpha\ell_0}$, there exists a C^1-map $\xi \mapsto n_\xi \in C^{\mathrm{sym}}(\mathbb{R}; [-1, 1])$, $\xi > \ell_0$, with the following properties. For some $\delta_0 > 0$ and any $\kappa > 0$, there is $c_0 = c_0(\kappa) > 0$ such that, for any $\xi \in \Gamma_{\ell_0,\kappa}$,

$$\|n_\xi(x) - \bar{m}_\xi(x)\|_\infty \leqslant c_0 e^{-\alpha\xi}, \tag{2.13}$$

$$\|\partial_\xi n_\xi(x) - \bar{m}'_\xi(x)\|_\infty \leqslant c_0 e^{-\alpha\xi}, \tag{2.14}$$

$$\|f(n_\xi) - V(\xi)\, \partial_\xi n_\xi\|_\infty \leqslant c_0 e^{-(2\alpha+\delta_0)\,\xi}, \tag{2.15}$$

with

$$V(\xi) \doteq -\mu K e^{-2\alpha\xi} + 2m_\beta\, \mu h \tag{2.16}$$

and α, μ, and K as in (2.6), (2.8), and (2.10) respectively.

This theorem will be proved in Section 8 where the function n_ξ is explicitly given in Definition 8.2. For any $\ell \geqslant \ell_0$ and $\kappa > 0$ we refer to $\mathscr{M}_{\ell,\kappa} \doteq \{n_\xi : \xi \in \Gamma_{\ell,\kappa}\}$ as the *quasi-invariant manifold*: by (2.15), $f(n_\xi)$, $n_\xi \in \mathscr{M}_{\ell,\kappa}$, is approximately tangent to $\mathscr{M}_{\ell,\kappa}$ and very small.

Theorem 2.2 and Proposition 2.3 below are consequences of the spectral analysis given in ref. 9, details are given in Section 8.

Proposition 2.3. Let n_ξ, $\xi \in \Gamma_{\ell_0,\kappa}$, be as in Theorem 2.2 and, recalling (2.3), set $L_\xi \doteq L_{n_\xi+h}$, $p_\xi \doteq p_{n_\xi+h}$. For any $\kappa > 0$ there are $\ell_1 \geqslant \ell_0$, $c_1 > 1$, and $\omega_1 > 0$ such that, for any $h < e^{-2\alpha\ell_1}$ and $\xi \in \Gamma_{\ell_1,\kappa}$, the following holds.

There exist an eigenvalue $\lambda_\xi > 0$ and strictly positive right and left eigenfunctions $v_\xi, v_\xi^* \in C_0^{\rm sym}(\mathbb{R})$ so that:

$$v_\xi^* = p_\xi v_\xi, \qquad \|e^{L_\xi t}\|_\infty \leqslant c_1 e^{\lambda_\xi t} \qquad \forall t \geqslant 0. \tag{2.17}$$

Furthermore letting $\alpha' = \alpha'(\xi) \doteq \alpha - c_1 e^{-2\alpha\xi}$ we have:

$$c_1^{-1} e^{-2\alpha\xi} \leqslant \lambda_\xi \leqslant c_1 e^{-2\alpha\xi} \qquad v_\xi(x) \leqslant c_1 e^{-\alpha' |\xi - x|} \qquad \forall x \in \mathbb{R}_+. \tag{2.18}$$

Moreover, recalling (2.8) and setting $\tilde{m}_\xi(x) \doteq \sqrt{\mu}\, \bar{m}(\xi - x)$,

$$|v_\xi(x) - \tilde{m}'_\xi(x)| \leqslant c_1 e^{-2\alpha\xi + \alpha |\xi - x|} \xi^4 \qquad \text{for} \quad |\xi - x| \leqslant \xi/2. \tag{2.19}$$

Assume v_ξ and v_ξ^* are normalized so that:

$$\int_0^\infty dx \, \frac{v_\xi(x)^2}{p_\xi(x)} \equiv \int_0^\infty dx \, v_\xi^*(x)\, v_\xi(x) = 1 \tag{2.20}$$

and define the linear functional π_ξ on $L_\infty^{\rm sym}(\mathbb{R})$ as

$$\pi_\xi(\psi) \doteq \int_0^\infty dx \, v_\xi^*(x)\, \psi(x). \tag{2.21}$$

Then, for any $\psi \in L_\infty^{\rm sym}(\mathbb{R})$ and for any $w \in L_\infty^{\rm sym}(\mathbb{R})$ such that $\pi_\xi(w) = 0$,

$$|\pi_\xi(\psi)| \leqslant c_1 \, \|\psi\|_\infty, \qquad \|e^{L_\xi t} w\|_\infty \leqslant c_1 e^{-\omega_1 t} \, \|w\|_\infty \qquad \forall t \geqslant 0. \tag{2.22}$$

We will often use that, as a consequence of the above proposition, for some constant $\hat{c}_1 = \hat{c}_1(\kappa)$, and for all w such that $\pi_\xi(w) = 0$,

$$\|L_\xi^{-1}\|_\infty \leqslant \hat{c}_1 \lambda_\xi^{-1}, \qquad \|L_\xi^{-1} w\|_\infty \leqslant \frac{c_1}{\omega_1} \, \|w\|_\infty. \tag{2.23}$$

Fermi Coordinates. We introduce tubular coordinates in a neighborhood of $\mathscr{M}_{\ell,\kappa}$ in the following way. The estimates (2.14) and (2.19) suggest that for ξ large enough the vectors v_ξ and $\partial_\xi n_\xi$ are almost parallel. Therefore we can use the projection (2.21) to define a transverse direction to $\mathscr{M}_{\ell,\kappa}$. To this end we shall need the following lemma, proved in Section 8.

Lemma 2.4. There are $\delta_1 > 0$ and $\alpha_1 > \alpha$ and, for any $\kappa > 0$, there are $\ell_2 \geqslant \ell_1$ and $c_2 > 0$ such that, for any $h < e^{-2\alpha\ell_2}$ and $\xi \in \Gamma_{\ell_2,\kappa}$, the following holds.

$$|\partial_\xi v_\xi(x) - \tilde{m}''_\xi(x)| \leqslant c_2 e^{-\alpha\xi}\xi^4 \qquad \text{for} \quad |\xi - x| \leqslant \xi/2, \tag{2.24}$$

$$|\partial_\xi v_\xi(x)| \leqslant c_2 \xi^2 v_\xi(x) \qquad \forall x \in \mathbb{R}, \qquad \left|\frac{d\lambda_\xi}{d\xi}\right| \leqslant c_2 e^{-\alpha_1 \xi}. \tag{2.25}$$

Moreover, for any $\psi \in L_\infty^{\mathrm{sym}}(\mathbb{R})$,

$$|\partial_\xi \pi_\xi(\psi)| \leqslant c_2 \|\psi\|_\infty, \qquad \partial_\xi \pi_\xi(\psi) \doteq \int_0^\infty dx\, \partial_\xi v_\xi^*(x)\, \psi(x). \tag{2.26}$$

Finally:

$$\|\partial_\xi n_\xi - \pi_\xi(\partial_\xi n_\xi)\, v_\xi\|_\infty \leqslant c_2 e^{-\delta_1 \xi}, \tag{2.27}$$

$$|\pi_\xi(\partial_\xi n_\xi)^{-1} - \sqrt{\mu}| \leqslant c_2 e^{-\delta_1 \xi}, \tag{2.28}$$

$$|\partial_\xi \pi_\xi(\partial_\xi n_\xi)| \leqslant c_2 e^{-\delta_1 \xi}. \tag{2.29}$$

Given $\varepsilon > 0$, $\kappa > 0$, $\ell \geqslant \ell_1$, and $h < e^{-2\alpha\ell}$, let

$$\mathscr{B}_{\ell,\kappa,\varepsilon} \doteq \{m \in L_\infty^{\mathrm{sym}}(\mathbb{R}) : \inf_{\xi \in \Gamma_{\ell,\kappa}} \|m - n_\xi\|_\infty < \varepsilon\}. \tag{2.30}$$

A standard application of the Implicit Function Theorem implies the following result.

Theorem 2.5. For any $\kappa > 0$ let ℓ_2 be as in Lemma 2.4. Then there are $\varepsilon_0 > 0$, $\bar{c} \geqslant 1$, $\bar{\kappa} > \kappa$, and $\ell_3 > \bar{\ell} \geqslant \ell_2$ such that, for any $h < e^{-2\alpha\ell_3}$, there exists a C^1-map $\Xi\colon \mathscr{B}_{\ell_3,\kappa,\varepsilon_0} \to \Gamma_{\bar{\ell},\bar{\kappa}}$ such that, for $\xi = \Xi(m)$,

$$\pi_\xi(m - n_\xi) = 0, \qquad \|m - n_\xi\|_\infty \leqslant \bar{c} \inf\{\|m - n_\zeta\|_\infty \colon \zeta \in \Gamma_{\ell_3,\kappa}\}. \tag{2.31}$$

Moreover, if $\xi_0 \in \Gamma_{\ell_3,\kappa}$ and $\|m - n_{\xi_0}\|_\infty < \varepsilon_0$, then:

$$|\xi - \xi_0| \leqslant \bar{c}\, \|m - n_{\xi_0}\|_\infty. \tag{2.32}$$

The proof, which is sketched at the end of Section 8, is a standard application of the Contraction Mapping Principle, see also the analogous result in ref. 5. As a corollary we get the existence of tubular coordinates. That is the following holds. Let

$$\bar{\mathscr{S}}_{\ell,\kappa,\varepsilon} \doteq \{(\xi, \varphi) \in \Gamma_{\ell,\kappa} \times L_\infty^{\mathrm{sym}}(\mathbb{R}) : \pi_\xi(\varphi) = 0,\ \|\varphi\|_\infty < \varepsilon\}. \tag{2.33}$$

For $(\xi, \varphi) \in \bar{\mathscr{S}}_{\ell,\kappa,\varepsilon}$ define $M(\xi, \varphi) \doteq n_\xi + \varphi$ and $\mathscr{S}_{\ell,\kappa,\varepsilon} \doteq M(\bar{\mathscr{S}}_{\ell,\kappa,\varepsilon})$. Then for any $h < e^{-2\alpha\ell_3}$, the map $M\colon \bar{\mathscr{S}}_{\ell_3,\kappa,\varepsilon_0} \to \mathscr{S}_{\ell_3,\kappa,\varepsilon_0}$ is differentiable, one to one, and $\Xi(M(\xi, \varphi)) = \xi$. Moreover $\mathscr{S}_{\ell_3, \kappa, \varepsilon_0}$ is open in $L_\infty^{\mathrm{sym}}(\mathbb{R})$. We thus have existence of tubular coordinates in a neighborhood of the manifold $\mathscr{M}_{\ell, \kappa}$ whose size is not vanishing as $h \downarrow 0$.

Equations of Motion. Consider a solution to (2.1), $S_t(m)$, $t \in [0, T]$, $T > 0$, which lies in the tubular neighborhood $\mathscr{S}_{\ell_3, \kappa, \varepsilon_0}$ with ℓ_3 and ε_0 as above. Then we can decompose $S_t(m) = n_{\xi(t)} + \varphi(t)$. By differentiating the identity $\pi_\xi(\varphi) = 0$, from (2.1) we obtain the equations of motion in the Fermi coordinates (ξ, φ):

$$[\pi_\xi(\partial_\xi n_\xi) - \partial_\xi \pi_\xi(\varphi)]\, \dot{\xi} = \pi_\xi(f(n_\xi + \varphi)), \tag{2.34}$$

$$\dot{\varphi} = f(n_\xi + \varphi) - \partial_\xi n_\xi \dot{\xi}, \tag{2.35}$$

where $(\dot{\xi}, \dot{\varphi})$ denotes the time derivative of (ξ, φ). Observe that, from Lemma 2.4, by choosing ε small enough (depending on κ), the coefficient of $\dot{\xi}$ in (2.34) is nonzero for any (ξ, φ) in the tubular neighborhood $\mathscr{S}_{\ell_3, \kappa, \varepsilon}$.

Given $\kappa > 0$ and $\ell > \ell_3$ such that $e^{-3\alpha\ell/2} < \varepsilon_0$, we introduce the *thin channels* $\mathscr{Z}_{\ell, \kappa, Q}$, $Q \geqslant 1$, defined by:

$$\mathscr{Z}_{\ell, \kappa, Q} \doteq \{m = n_\xi + \varphi \in \mathscr{S}_{\ell_3, \kappa, \varepsilon} : \xi \in \Gamma_{\ell, \kappa},\ \|\varphi\|_\infty \leqslant Q^{-1} e^{-3\alpha\xi/2}\}. \tag{2.36}$$

Then:

Theorem 2.6. For any $\kappa > 0$ there are $\varepsilon_1 \in (0, \varepsilon_0]$, $\ell_4 \geqslant \ell_3$, $Q \geqslant 1$, and $v > 0$, such that, for any $h < e^{-2\alpha\ell_4}$, the following holds. If $m \in \mathscr{S}_{\ell_4, \kappa, \varepsilon_1}$ then, as long as $S_t(m) = n_{\xi(t)} + \varphi(t) \in \mathscr{S}_{\ell_4, \kappa, \varepsilon_0}$, one has:

$$\|\varphi(t)\|_\infty \leqslant e^{-3\alpha\xi(t)/2} + (Q\, \|\varphi(0)\|_\infty - e^{-3\alpha\xi(0)/2})\, e^{-vt}. \tag{2.37}$$

We may assume ε_1 so small and ℓ_4 so large that $(1+Q)\,\varepsilon_1 < \varepsilon_0$ and $e^{-3\alpha\ell_4/2} < \varepsilon_1$. By (2.37) we thus see that the tube $\mathscr{S}_{\ell_4, \kappa, \varepsilon_1}$ is exponentially attracted toward the channel $\mathscr{Z}_{\ell_4, \kappa, 1}$. Moreover, for $m \in \mathscr{Z}_{\ell, \kappa, Q}$ there are only two possibilities: either $S_t(m)$ stays forever in $\mathscr{Z}_{\ell, \kappa, 1}$, or there is a finite time t^* for which $\xi(t^*)$ belongs to the boundary of $\Gamma_{\ell_4, \kappa}$. The dynamics in the channel $\mathscr{Z}_{\ell_4, \kappa, 1}$ is essentially reduced to the motion of the coordinate $\xi(t)$, for which we have an explicit formula:

Theorem 2.7. For any $\kappa > 0$ there is $\delta^* > 0$ such that, for any $\ell > \ell_3$ sufficiently large, as long as $S_t(m) = n_{\xi(t)} + \varphi(t) \in \mathscr{Z}_{\ell, \kappa, 1}$, one has (recall the definition (2.16)):

$$\dot{\xi} = V(\xi) + O(e^{-(2\alpha + \delta^*)\,\xi}). \tag{2.38}$$

Theorems 2.6 and 2.7 will be proved in Section 3.

The Critical Droplet. Our next result, proved in Section 4, concerns the existence and uniqueness of the critical droplet in a neighborhood of $\mathcal{M}_{\ell,\kappa}$.

Theorem 2.8. There is κ_0 such that the following holds. For any $\kappa \geqslant \kappa_0$ there are $\varepsilon_2 \in (0, \varepsilon_0]$, $\ell_5 \geqslant \ell_3$, and $h_0 < e^{-2\alpha\ell_5}$ such that, for any $h \in (0, h_0]$, the equation $f(m) = 0$ in $\mathcal{S}_{\ell_5,\kappa,\varepsilon_2}$ has a unique solution $m = n_{\bar{\xi}} + \bar{\varphi}$. Moreover:

$$\lim_{h \downarrow 0} e^{2\alpha\bar{\xi}} h = (2m_\beta)^{-1} K, \qquad \lim_{h \downarrow 0} e^{2\alpha\bar{\xi}} \|\bar{\varphi}\|_\infty = 0, \tag{2.39}$$

with K as in (2.10).

From (2.39), (2.13), and (2.9) this solution to $f(m) = 0$ has to coincide with the critical droplet of Theorem 2.1, i.e., $q = n_{\bar{\xi}} + \bar{\varphi}$.

We have a detailed information on the spatial structure of the critical droplet, which is the content of the following proposition, proved in Section 5.

Proposition 2.9. Given $\beta > 1$, there is $h^* \in (0, h_0]$ (h_0 as in Theorem 2.1) such that for any $h \in (0, h^*]$ the bump $q(x)$ is a strictly decreasing function on $\mathbb{R}_+$ (actually $q'(x) < 0$ for all $x > 0$). Moreover, letting $\gamma > 0$ be such that

$$\beta[1 - (m^-_{\beta,h})^2] \int dz\, J(z)\, e^{-\gamma z} = 1, \tag{2.40}$$

and ξ_q be the (unique) positive zero of $q(x)$, there are $A > 0$, $\delta > 0$, and $C > 0$ so that, for all $x \geqslant \xi_q$,

$$\begin{aligned} &|q(x) - (m^-_{\beta,h} + Ae^{-\gamma(x-\xi_q)})| + |q'(x) + \gamma A e^{-\gamma(x-\xi_q)}| + |q''(x) - \gamma^2 A e^{-\gamma(x-\xi_q)}| \\ &\quad \leqslant C e^{-(\gamma+\delta)(x-\xi_q)}. \end{aligned} \tag{2.41}$$

Finally, as $h \downarrow 0$, A, δ, and C remain strictly positive and bounded, while $\xi_q \to +\infty$.

The Invariant Manifold. The behavior of the dynamics around q follows from the spectral properties of the linear operator $L \doteq L_{q+h}$. Since q satisfies (1.2) for any $\psi \in L^{\mathrm{sym}}_\infty(\mathbb{R})$,

$$L\psi = -\psi + pJ * \psi, \qquad p(x) \doteq \beta[1 - q(x)^2]. \tag{2.42}$$

Given $\zeta \in \mathbb{R}$, we introduce the normed spaces $X_\zeta \doteq \{w: \mathbb{R} \to \mathbb{R}$ measurable and symmetric : $\|w\|_\zeta < \infty\}$ where

$$\|w\|_\zeta \doteq \sup_{x \in \mathbb{R}_+} e^{\zeta x} |w(x)|. \tag{2.43}$$

The following proposition follows from the results proved in refs. 9 and 10 and for the reader convenience we give detailed references on where the proofs can be found in Section 8.

Proposition 2.10. Given $\beta > 1$ let h^* be as in Proposition 2.9. Then there are constants $C_0 > 1$ and $C_1 > 0$ such that for any $h \in (0, h^*]$ the following holds. There is an eigenvalue $\lambda > 0$ and strictly positive right and left eigenfunctions $v, v^* \in C_0^{\mathrm{sym}}(\mathbb{R})$ so that $v^*(x) = p(x)\, v(x)$ for all $x \in \mathbb{R}$ and

$$\frac{h}{C_0} \leqslant \lambda \leqslant C_0 h. \tag{2.44}$$

Furthermore there is a unique $\gamma_v > \gamma > 0$, (γ as in (2.40)) such that:

$$\beta(1-(m^-_{\beta,h})^2) \int dz\, J(z)\, e^{-\gamma_v z} = 1+\lambda, \tag{2.45}$$

and there is $M_v > 0$ for which

$$\lim_{x \to +\infty} e^{\gamma_v x} v(x) = M_v. \tag{2.46}$$

Moreover, for any $\zeta \leqslant \gamma_v$ and $t \geqslant 0$,

$$\|e^{Lt} w\|_\zeta \leqslant C_1 e^{\lambda t} \|w\|_\zeta \qquad \forall w \in X_\zeta. \tag{2.47}$$

Assume v and v^* are normalized as in (2.20) (with p_ξ replaced by p) and define the linear functional π on X_ζ, $\zeta < \gamma_v$, as in (2.21). Then there is $\omega > 0$ so that, for any $w \in L_\infty^{\mathrm{sym}}(\mathbb{R})$ such that $\pi(w) = 0$ and $t \geqslant 0$,

$$\|e^{Lt} w\|_\infty \leqslant C_1 e^{-\omega t} \|w\|_\infty, \qquad \frac{1}{C_0} \leqslant \omega \leqslant C_0. \tag{2.48}$$

Finally, for ξ^* as in (2.9),

$$\lim_{h \downarrow 0} \sup_{x \leqslant 0} |v(x) - \sqrt{\mu}\, \bar{m}'(x+\xi^*)| = 0. \tag{2.49}$$

From the above spectral estimates it follows the existence of two distinct, one dimensional manifolds $\mathscr{W}_{\pm} \subset C^{\text{sym}}(\mathbb{R}; [-1, 1])$. We give the precise statement in the next theorem, proved in Section 6.

Theorem 2.11. Given $\beta > 1$ let h^* be as in Proposition 2.9. Then, for any $h \in (0, h^*]$, there are two distinct, one dimensional manifolds $\mathscr{W}_{\pm} \subset C^{\text{sym}}(\mathbb{R}; [-1, 1])$, such that:

$$\mathscr{W}_{\pm} = \{m_s^{\pm} : s \in \mathbb{R}\}, \qquad S_t(m_s^{\pm}) = m_{s+t}^{\pm}, \qquad \forall s \in \mathbb{R}, \quad \forall t \geqslant 0, \tag{2.50}$$

$$\lim_{s \to -\infty} \|m_s^{\pm} - q\|_\infty = 0. \tag{2.51}$$

Moreover, for μ as in (2.8),

$$\lim_{s \to -\infty} e^{-\lambda s} \left\| \frac{dm_s^{\pm}}{ds} \mp \lambda e^{\lambda s} \frac{1}{\sqrt{\mu}} v \right\|_\infty = 0. \tag{2.52}$$

Finally, for any $s \in \mathbb{R}$, the (symmetric) functions $m_s^{\pm}$ are nonincreasing on $\mathbb{R}_+$ and satisfy:

$$m_{\beta,h}^- \leqslant m_s^-(x) \leqslant q(x) \leqslant m_s^+(x) \leqslant m_{\beta,h}^+ \qquad \forall x \in \mathbb{R}. \tag{2.53}$$

Thus the one dimensional manifolds $\mathscr{W}_{\pm}$ originates at $s = -\infty$ from q and are time invariant. Each one of them is therefore described by a single orbit of S_t with time going from $-\infty$ to $+\infty$. The two orbits are denoted by $m_s^{\pm}$ and the parameter s is identified with time. Of course the origin of time is arbitrary and this can be exploited to fix up the constants in such a way that (2.52) holds, we refer to Section 6 for details on this point.

By integrating (2.52) from $-\infty$ to s we get $m_s^{\pm} \approx q \pm e^{\lambda s} v / \sqrt{\mu}$. Then, by (2.9) and (2.49), $m_s^{\pm}(x) \approx \bar{m}(x + \xi^* \pm e^{\lambda s})$ for h small and $x \leqslant 0$. By symmetry the result extends to $x \geqslant 0$. Thus the points in a neighborhood of q that are in $\mathscr{W}_+$ are "droplets longer" than q while those in $\mathscr{W}_-$ are shorter. Their length changes at the exponential rate λ, which is therefore the Lyapunov exponent at q, with $\mathscr{W}_{\pm}$ the corresponding unstable manifolds. Since $\lambda \approx h$, for h small, there is a *dormant instability*, in the sense that for small h (which is the case of interest in metastability) even though ultimately unstable, the bump q seems in fact stable for very long times (of order h^{-1}).

By (2.39) the critical droplet q belongs to the interior of the channel $\mathscr{L}_{\ell_4, \kappa, Q}$, for κ large and h small enough. Then, according to Theorem 2.6, a large part $\overline{\mathscr{W}}$ of the invariant manifold $\mathscr{W}$ is contained in $\mathscr{L}_{\ell_4, \kappa, 1}$: it thus consists of droplets with two well-separated transition layers whose locations evolve according to Eq. (2.38).

Our last result concerns the global structure of the invariant manifold $\mathscr{W}$. More precisely we show that the sub-critical and super-critical droplets will eventually go to the metastable and stable phase respectively: this is the content of the next theorem proved in Section 7.

Theorem 2.12. Given $\beta > 1$ there is $h^\dagger \in (0, h^*]$ (h^* as in Proposition 2.9) such that, for any $h \in (0, h^\dagger]$,

$$\lim_{s \to +\infty} \|m_s^- - m_{\beta,h}^-\|_\infty = 0, \tag{2.54}$$

$$\lim_{s \to +\infty} m_s^+(x) = m_{\beta,h}^+ \qquad \forall x \in \mathbb{R}. \tag{2.55}$$

A Notation Warning. In Sections 3, 4, and 8 we shall denote by $C = C(\kappa)$ a generic positive function of κ whose numerical value may change from line to line.

3. LOCAL ATTRACTIVENESS

We start with a preliminary lemma.

Lemma 3.1. For any $\kappa > 0$ let ℓ_3 and ε_0 be as in Theorem 2.5. Then, there exists $G = G(\kappa)$ such that, for any $h < e^{-2\alpha\ell_3}$, $N > 0$, and $m = n_\xi + \varphi \in \mathscr{S}_{\ell_3, \kappa, \varepsilon_0}$, one has:

$$\|S_t(m) - n_\xi\|_\infty \leqslant (c_1 e^{-\omega_1 t} + N\,\|\varphi\|_\infty)\,\|\varphi\|_\infty + N e^{-2\alpha\xi} \qquad \forall t < T_{m,N}, \tag{3.1}$$

with c_1 and ω_1 as in Proposition 2.3 and

$$T_{m,N} \doteq \min\left\{ e^{2\alpha\xi}; \frac{N}{G[1 + N^2(e^{-2\alpha\xi} + \|\varphi\|_\infty^2)]} \right\}. \tag{3.2}$$

Proof. Recalling that $L_\xi \doteq L_{n_\xi + h}$, the function $\psi_t \doteq S_t(m) - n_\xi$, $\psi_0 = \varphi$, solves

$$\psi_t = e^{L_\xi t}\varphi + \int_0^t ds\, e^{L_\xi (t-s)}[f(n_\xi + \psi_s) - L_\xi \psi_s].$$

By (2.17) and (2.5) we then get:

$$\|\psi_t - e^{L_\xi t}\varphi\|_\infty \leqslant c_1 \int_0^t ds\, e^{\lambda_\xi (t-s)}(\|f(n_\xi)\|_\infty + k_2\,\|\psi_s\|_\infty^2).$$

By (2.22), since $\pi_\xi(\varphi)=0$, $\|\psi_s\|_\infty \leqslant c_1 \|\varphi\|_\infty + \|\psi_s - e^{L_\xi s}\varphi\|_\infty$. Moreover, by (2.12), (2.15), and (2.16), $\|f(n_\xi)\|_\infty \leqslant Ce^{-2\alpha\xi}$. Finally, by (2.18), if $t \leqslant e^{2\alpha\xi}$ then $\lambda_\xi t \leqslant c_1$. We conclude that there exists $G = G(\kappa) > 0$ such that:

$$\|\psi_t - e^{L_\xi t}\varphi\|_\infty \leqslant G \int_0^t ds(e^{-2\alpha\xi} + \|\varphi\|_\infty^2 + \|\psi_s - e^{L_\xi t}\varphi\|_\infty^2) \qquad \forall t \leqslant e^{2\alpha\xi}. \tag{3.3}$$

Given $N > 0$, fix $\tau < T_{m,N}$ and let $T \leqslant \tau$ be the first time when the inequality

$$\|\psi_t - e^{L_\xi t}\varphi\|_\infty \leqslant N(e^{-2\alpha\xi} + \|\varphi\|_\infty^2) \tag{3.4}$$

becomes an equality. Then, by (3.3),

$$N(e^{-2\alpha\xi} + \|\varphi\|_\infty^2) \leqslant N(e^{-2\alpha\xi} + \|\varphi\|_\infty^2)\frac{\tau}{T_{m,N}} < N(e^{-2\alpha\xi} + \|\varphi\|_\infty^2).$$

We have thus reached a contradiction and hence (3.4) is valid for all $t \leqslant \tau$. By (2.22) and (3.4) the inequality in (3.1) holds for any $t \leqslant \tau$. The lemma is thus proved. ∎

Proof of Theorem 2.6. We fix the parameter N of Lemma 3.1 such that

$$4\bar{c}c_1 e^{3\alpha\bar{c}\varepsilon_0/2} e^{-\omega_1 T_N} = 1, \qquad T_N \doteq \frac{N}{2G}, \tag{3.5}$$

with c_1, ω_1 as in Proposition 2.3, $\bar{c}$, ε_0 as in Theorem 2.5, and G as in Lemma 3.1. We next define:

$$Q \doteq \tfrac{1}{2}(1 + e^{\omega_1 T_N}) = \tfrac{1}{2} + 2\bar{c}c_1 e^{3\alpha\bar{c}\varepsilon_0/2}, \qquad b \doteq \tfrac{1}{2}e^{-3\alpha\bar{c}\varepsilon_0/2}. \tag{3.6}$$

(note that $Q \geqslant 1$ and $b < 1/2$). We finally choose $\ell_4 > \ell_3$ large enough and $\varepsilon_1 \in (0, \varepsilon_0]$ small enough that:

$$N^2(e^{-2\alpha\ell_4} + \varepsilon_1^2) < 1, \qquad Ne^{-2\alpha\ell_4} < 2G, \qquad c_1\varepsilon_1 + N(\varepsilon_1^2 + e^{-2\alpha\ell_4}) < \varepsilon_0, \tag{3.7}$$

$$\bar{c}N\varepsilon_1 < b/2, \qquad \bar{c}Ne^{-\alpha\ell_4/2} < b/Q, \qquad e^{-3\alpha\ell_4/2} < \varepsilon_1/2. \tag{3.8}$$

Let now $m = n_\xi + \varphi \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_1}$. By the first and the second inequality in (3.7) we have $T_{m,N} > T_N$ for all $m \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_1}$ with $T_{m,N}$ as in (3.2). By (3.1) and the third inequality in (3.7) we thus get $\|S_t(m) - n_\xi\|_\infty < \varepsilon_0$ for all $t \in [0, T_N]$.

Then, by Theorem 2.5, the function $\xi(t)=\Xi(S_t(m))$ is well defined for all $t\in[0,T_N]$ and

$$|\xi(t)-\xi|\leqslant\bar{c}\,\|S_t(m)-n_\xi\|_\infty\leqslant\bar{c}\varepsilon_0, \tag{3.9}$$

$$\begin{aligned}\|\varphi(t)\|_\infty&\leqslant\bar{c}\,\|S_t(m)-n_\xi\|_\infty\\&\leqslant\bar{c}(c_1e^{-\omega_1 t}+N\varepsilon_1)\,\|\varphi\|_\infty+\bar{c}Ne^{-\alpha\ell_4/2}e^{-3\alpha\xi/2},\end{aligned} \tag{3.10}$$

where $\varphi(t)\doteq S_t(m)-n_{\xi(t)}$ and we again used (3.1) for the last bound in (3.10). By (3.5), (3.8), and (3.10) it easily follows that:

$$\|\varphi(t)\|_\infty\leqslant\frac{b}{2}(1+e^{\omega_1(T_N-t)})\,\|\varphi\|_\infty+\frac{b}{Q}\,e^{-3\alpha\xi/2}.$$

Moreover, by (3.9), $2be^{-3\alpha\xi/2}\leqslant e^{-3\alpha\xi(t)/2}$ for all $t\in[0,T_N]$. Then, recalling the definition (3.6) of $Q\geqslant 1$, we conclude that:

$$\|\varphi(t)\|_\infty\leqslant b(Q\,\|\varphi\|_\infty-e^{-3\alpha\xi/2})+e^{-3\alpha\xi(t)/2}\qquad\forall t\in[0,T_N], \tag{3.11}$$

$$\|\varphi(T_N)\|_\infty\leqslant\frac{b}{Q}(Q\,\|\varphi\|_\infty-e^{-3\alpha\xi/2})+\frac{1}{Q}\,e^{-3\alpha\xi(T_N)/2}. \tag{3.12}$$

We have thus proved that if $m\in\mathscr{S}_{\ell_4,\kappa,\varepsilon_1}$ then $S_t(m)\in\mathscr{B}_{\ell_4,\kappa,\varepsilon_0}$ for all $t\in[0,T_N]$ and $\varphi(t)=S_t(m)-n_{\xi(t)}$ satisfies the bounds (3.11) and (3.12). It follows that $S_t(m)$ may leave $\mathscr{S}_{\ell_4,\kappa,\varepsilon_0}$ at a given time $t^*\in[0,T_N]$ only because $\xi(t^*)$ belongs to the boundary of $\Gamma_{\ell_4,\kappa}$. If this is not the case, since the third inequality in (3.8) implies $\|\varphi(T_N)\|_\infty<\varepsilon_1$, then $S_{T_N}(m)\in\mathscr{S}_{\ell_4,\kappa,\varepsilon_1}$ and we can repeat the same analysis for the solution in the interval $[T_N,2T_N]$. Let n be the largest integer such that $S_t(m)$ leaves $\mathscr{S}_{\ell_4,\kappa,\varepsilon_0}$ at $t^*\geqslant t_n\doteq nT_N$, setting $n=+\infty$ if $S_t(m)\in\mathscr{S}_{\ell_4,\kappa,\varepsilon_0}$ forever. Then, for any integer $k\leqslant n$ we iterate (3.12) getting

$$\|\varphi(t_k)\|_\infty\leqslant\frac{b^k}{Q}(Q\,\|\varphi(0)\|_\infty-e^{-3\alpha\xi(0)/2})+\frac{1}{Q}\,e^{-3\alpha\xi(t_k)/2},\qquad t_k\doteq kT_N,$$

from which, using (3.11) with $\varphi=\varphi(t_k)$, we finally obtain:

$$\|\varphi(t)\|_\infty\leqslant b^{k+1}(Q\,\|\varphi(0)\|_\infty-e^{-3\alpha\xi(0)/2})+e^{-3\alpha\xi(t)/2}\qquad\forall t\in[t_k,t_{k+1}],$$

which implies (2.37) with $\nu=-(\log b)/T_N$. The theorem is thus proved. ▮

Proof of Theorem 2.7. Recalling (2.4),

$$f(n_\xi+\varphi)=f(n_\xi)+L_\xi\varphi+R_\xi[\varphi], \tag{3.13}$$

$$R_\xi[\varphi]=\beta^2(J*\varphi)^2\int_0^1 ds(1-s)\tanh''\{\beta[J*(n_\xi+s\varphi)+h]\}, \tag{3.14}$$

so that, since $\pi_\xi(L_\xi\varphi)=\lambda_\xi\pi_\xi(\varphi)=0$, Eq. (2.34) becomes:

$$\dot{\xi}=\frac{\pi_\xi(f(n_\xi))+\pi_\xi(R_\xi[\varphi])}{\pi_\xi(\partial_\xi n_\xi)-\partial_\xi\pi_\xi(\varphi)}.$$

By (2.5), (2.15), (2.22), (2.26), and the definition of $\mathscr{L}_{\ell,\kappa,1}$, (2.38) follows for a suitable $\delta^*>0$. ∎

4. THE CRITICAL DROPLET

In this section we prove Theorem 2.8. Recalling (2.34) and (2.35) we have that $m=n_\xi+\varphi\in\mathscr{S}_{\ell_3,\kappa,\varepsilon_0}$ is a solution to $f(m)=0$ iff (ξ,φ) solve:

$$f(n_\xi+\varphi)=0,\qquad \pi_\xi(f(n_\xi+\varphi))=0,\qquad \pi_\xi(\varphi)=0.$$

We rewrite the above equations as follows. For $\xi\in\Gamma_{\ell_3,\kappa}$ we define the projection T_ξ acting on $L_\infty^{\mathrm{sym}}(\mathbb{R})$ by:

$$T_\xi\psi=\psi-\pi_\xi(\psi)\,v_\xi. \tag{4.1}$$

Observe that by (2.18) and (2.22) there is C so that:

$$\|T_\xi\psi\|_\infty\leqslant C\,\|\psi\|_\infty\qquad \forall\psi\in L_\infty^{\mathrm{sym}}(\mathbb{R}). \tag{4.2}$$

Then, using the expansion (3.13) and the fact that $\pi_\xi(L_\xi\varphi)=0$ we get that $f(n_\xi+\varphi)=0$ iff (ξ,φ) solve:

$$L_\xi\varphi+T_\xi(f(n_\xi)+R_\xi[\varphi])=0, \tag{4.3}$$

$$\pi_\xi(f(n_\xi)+R_\xi[\varphi])=0. \tag{4.4}$$

We now proceed in the following way: we first show in Lemma 4.1 later, that there exists a unique graph $\xi\mapsto\varphi_\xi$ in a tubular neighborhood $\mathscr{S}_{\ell,\kappa,\varepsilon}$ such that φ_ξ is solution to (4.3). Replacing φ by φ_ξ in (4.4) the problem reduces to study an equation in the real variable ξ.

Lemma 4.1. For any $\kappa > 0$ there are $\varepsilon_2 \in (0, \varepsilon_0]$ and $\ell_6 \geqslant \ell_3$ such that, for any $h < e^{-2\alpha\ell_6}$ and $\xi \in \Gamma_{\ell_6,\kappa}$, the map $A_\xi\colon L_\infty^{\mathrm{sym}}(\mathbb{R}) \to L_\infty^{\mathrm{sym}}(\mathbb{R})$ defined by

$$A_\xi(\varphi) \doteq -L_\xi^{-1} T_\xi(f(n_\xi) + R_\xi[\varphi]) \tag{4.5}$$

is a contraction in $Y_{\varepsilon_2} \doteq \{\varphi \in L_\infty^{\mathrm{sym}}(\mathbb{R}) : \|\varphi\|_\infty \leqslant \varepsilon_2\}$.

Proof. From (2.5) and (2.23) we have:

$$\|A_\xi \varphi\|_\infty \leqslant \frac{c_1}{\omega_1} [\|T_\xi f(n_\xi)\|_\infty + k_2 \|\varphi\|_\infty^2].$$

We write $T_\xi f(n_\xi) = T_\xi(f(n_\xi) - V(\xi)\,\partial_\xi n_\xi) + V(\xi)\, T_\xi\, \partial_\xi n_\xi$. From (2.15) and (4.2) it follows that: $\|T_\xi(f(n_\xi) - V(\xi)\,\partial_\xi n_\xi)\|_\infty \leqslant Ce^{-(2\alpha+\delta_0)\,\xi}$. Since for $\xi \in \Gamma_{\ell_3,\kappa}$ and, if $h < e^{-2\alpha\ell_3}$, then $he^{-2\alpha\xi} \leqslant e^{2\alpha\kappa}$, from (2.27) we get: $|V(\xi)| \times \|T_\xi\, \partial_\xi n_\xi\|_\infty \leqslant Ce^{-(2\alpha+\delta_1)\,\xi}$. By the above estimates we conclude that there is $\delta > 0$ so that:

$$\|T_\xi f(n_\xi)\|_\infty \leqslant Ce^{-(2\alpha+\delta)\,\xi}. \tag{4.6}$$

Then, for any $\ell \geqslant \ell_3$, $h < e^{-2\alpha\ell}$, $\xi \in \Gamma_{\ell,\kappa}$, and $\varphi \in Y_\varepsilon$,

$$\|A_\xi(\varphi)\|_\infty \leqslant \frac{c_1}{\omega_1} [Ce^{-(2\alpha+\delta)\,\ell} + k_2\varepsilon^2]. \tag{4.7}$$

Moreover, recalling (3.14), there is $k_3 > 0$ for which, if $\varphi_1, \varphi_2 \in Y_\varepsilon$, then:

$$\|R_\xi[\varphi_1] - R_\xi[\varphi_2]\|_\infty \leqslant k_3\varepsilon(1+\varepsilon)\, \|\varphi_1 - \varphi_2\|_\infty$$

so that

$$\|A_\xi(\varphi_1) - A_\xi(\varphi_2)\|_\infty \leqslant \frac{c_1 k_3}{\omega_1}\, \varepsilon(1+\varepsilon)\, \|\varphi_1 - \varphi_2\|_\infty. \tag{4.8}$$

From (4.7) and (4.8) we conclude that, by choosing $\varepsilon_2 = 2Cc_1\omega_1^{-1} e^{-(2\alpha+\delta)\,\ell_6}$ and $\ell_6 \geqslant \ell_3$ large enough, A_ξ is a contraction in Y_{ε_2}. ∎

Proposition 4.2. For any $\kappa > 0$ there is $\ell_7 > \ell_6$ such that, for any $h < e^{-2\alpha\ell_7}$, the graph $\xi \to \varphi_\xi$ is differentiable and

$$\|\varphi_\xi\|_\infty \leqslant Ce^{-(2\alpha+\delta_2)\,\xi}, \qquad \|\partial_\xi \varphi_\xi\|_\infty \leqslant Ce^{-\delta_2\xi} \tag{4.9}$$

with $\delta_2 = \min\{\delta_0, \delta_1, 2\alpha\}$ (δ_0 and δ_1 as in Theorem 2.2 and Lemma 2.4 respectively).

Proof. Since φ_ξ is the fixed point of the map (4.5), the bound for $\|\varphi_\xi\|_\infty$ follows immediately from (4.6) (with $\delta_2 = \delta$). Moreover, since the r.h.s. of (4.5) is a continuous function of ξ and φ, the continuity of $\xi \mapsto \varphi_\xi$ is also straightforward.

We next study the equation for $\partial_\xi \varphi_\xi$ which is obtained by differentiating (4.3) (for $\varphi = \varphi_\xi$) w.r.t. ξ. We shall prove that for any $\kappa > 0$ there is $\ell_7 \geqslant \ell_6$ such that it has a unique solution for any $\xi \in \Gamma_{\ell,\kappa}$. By standard arguments this solution defines the derivative $\partial_\xi \varphi_\xi$.

Let $G_\xi \doteq -(f(n_\xi) + R_\xi[\varphi_\xi])$, so that φ_ξ solves $L_\xi \varphi_\xi = T_\xi G_\xi$. By differentiating (recall (2.26)) we get:

$$
\begin{aligned}
L_\xi\, \partial_\xi \varphi_\xi &+ \partial_\xi\, p_\xi J * \varphi_\xi \\
&= T_\xi\, \partial_\xi G_\xi - \pi_\xi(G_\xi)\, T_\xi\, \partial_\xi v_\xi - [\pi_\xi(G_\xi)\, \pi_\xi(\partial_\xi v_\xi) + \partial_\xi \pi_\xi(G_\xi)]\, v_\xi .
\end{aligned}
$$

By substituting $G_\xi = L_\xi \varphi_\xi + \pi_\xi(G_\xi)\, v_\xi$ in the term $\partial_\xi \pi_\xi(G_\xi)$ we have:

$$
\begin{aligned}
L_\xi\, \partial_\xi \varphi_\xi = &- \partial_\xi\, p_\xi J * \varphi_\xi + T_\xi\, \partial_\xi G_\xi - \pi_\xi(G_\xi)\, T_\xi\, \partial_\xi v_\xi \\
&- \{\pi_\xi(G_\xi)[\pi_\xi(\partial_\xi v_\xi) + \partial_\xi \pi_\xi(v_\xi)] + \partial_\xi \pi_\xi(L_\xi \varphi_\xi)\}\, v_\xi .
\end{aligned}
$$

Observing now that, by (2.20), $\pi_\xi(\partial_\xi v_\xi) + \partial_\xi \pi_\xi(v_\xi) = 0$ we get

$$
L_\xi\, \partial_\xi \varphi_\xi = T_\xi[\partial_\xi G_\xi - \partial_\xi\, p_\xi J * \varphi_\xi - \pi_\xi(G_\xi)\, \partial_\xi v_\xi)] + \lambda_\xi H(\xi)\, v_\xi, \tag{4.10}
$$

with

$$
\begin{aligned}
H(\xi) &= -\frac{1}{\lambda_\xi}\, [\pi_\xi(\partial_\xi\, p_\xi J * \varphi_\xi) + \partial_\xi \pi_\xi(L_\xi \varphi_\xi)] \\
&= -\frac{1}{\lambda_\xi} \int_0^\infty dx \left[\frac{\partial_\xi\, p_\xi}{p_\xi}\, v_\xi J * \varphi_\xi + \left(\frac{\partial_\xi v_\xi}{p_\xi} - \frac{\partial_\xi\, p_\xi}{p_\xi^2}\, v_\xi \right) L_\xi \varphi_\xi \right](x) \\
&= -\frac{1}{\lambda_\xi} \int_0^\infty dx\, \varphi_\xi(x) \left(J * \partial_\xi v_\xi + \frac{\partial_\xi\, p_\xi}{p_\xi^2}\, v_\xi - \frac{\partial_\xi v_\xi}{p_\xi} \right)(x).
\end{aligned} \tag{4.11}
$$

In the last line we used the definition of L_ξ and that, since $\partial_\xi v_\xi$, J, and φ_ξ are symmetric functions, then:

$$
\int_0^\infty dx\, \partial_\xi v_\xi(x)\, J * \varphi_\xi(x) = \int_0^\infty dx\, \varphi_\xi(x)\, J * \partial_\xi v_\xi(x).
$$

Now, by differentiating $L_\xi v_\xi = \lambda_\xi v_\xi$,

$$J * \partial_\xi v_\xi = \frac{d\lambda_\xi}{d\xi}\frac{v_\xi}{p_\xi} + \lambda_\xi \frac{\partial_\xi v_\xi}{p_\xi} - \frac{\partial_\xi p_\xi}{p_\xi} J * v_\xi + \frac{\partial_\xi v_\xi}{p_\xi},$$

so that

$$H(\xi) = -\frac{1}{\lambda_\xi}\int_0^\infty dx\, \varphi_\xi(x)\left(\frac{d\lambda_\xi}{d\xi}\frac{v_\xi}{p_\xi} + \lambda_\xi \frac{\partial_\xi v_\xi}{p_\xi} - \frac{\partial_\xi p_\xi}{p_\xi^2} L_\xi v_\xi\right)(x).$$

Using again $L_\xi v_\xi = \lambda_\xi v_\xi$ and observing the first term on the r.h.s. is proportional to $\pi_\xi(\varphi_\xi) = 0$, we finally obtain:

$$H(\xi) = -\int_0^\infty dx\, \varphi_\xi(x)\left(\frac{\partial_\xi v_\xi}{p_\xi} - \frac{\partial_\xi p_\xi}{p_\xi^2} v_\xi\right)(x). \tag{4.12}$$

Recalling now $L_\xi = Df|_{n_\xi}$ and (3.14), we have:

$$\partial_\xi G_\xi = -L_\xi\, \partial_\xi n_\xi + K_\xi + Z_\xi J * \partial_\xi \varphi_\xi, \tag{4.13}$$

where

$$K_\xi = -\beta^3 (J * \varphi_\xi)^2 \int_0^1 ds\, s(1-s) \tanh'''\{\beta[J * (n_\xi + s\varphi_\xi) + h]\}\, J * \partial_\xi n_\xi, \tag{4.14}$$

$$\begin{aligned} Z_\xi = {} & -2\beta^2 J * \varphi_\xi \int_0^1 ds(1-s) \tanh''\{\beta[J * (n_\xi + s\varphi_\xi) + h]\} \\ & - \beta^3 (J * \varphi_\xi)^2 \int_0^1 ds\, s(1-s) \tanh'''\{\beta[J * (n_\xi + s\varphi_\xi) + h]\}. \end{aligned} \tag{4.15}$$

Observing that $T_\xi L_\xi\, \partial_\xi n_\xi = L_\xi T_\xi\, \partial_\xi n_\xi$, from (4.10) and (4.13) we obtain $\partial_\xi \varphi_\xi$ by solving the linear equation $\psi = U_\xi \psi + g_\xi$, where:

$$\begin{aligned} U_\xi \psi &\doteq L_\xi^{-1} T_\xi Z_\xi J * \psi, \\ g_\xi &\doteq L_\xi^{-1} T_\xi [K_\xi - \partial_\xi p_\xi J * \varphi_\xi - \pi_\xi(G_\xi)\, \partial_\xi v_\xi] + H(\xi)\, v_\xi - T_\xi\, \partial_\xi n_\xi. \end{aligned}$$

From (2.23) and (4.15) $\|U_\xi\|_\infty \leqslant C\, \|\varphi_\xi\|_\infty$, hence, by using the first bound in (4.9), there is $\ell_7 \geqslant \ell_6$ such that $\|U_\xi\|_\infty \leqslant 1/2$ for $\xi \in \Gamma_{\ell,\kappa}$. Then for such values of ξ the equation has a unique solution $\partial_\xi \varphi_\xi = (1 - U_\xi)^{-1} g_\xi$. The continuity of $\xi \mapsto \partial_\xi \varphi_\xi$ follows from that of $\xi \mapsto U_\xi$ and $\xi \mapsto g_\xi$, which can be easily proved.

We are left with the second bound in (4.9). Since $\|\partial_\xi \varphi_\xi\|_\infty \leqslant 2\, \|g_\xi\|_\infty$ we have only to prove an analogous estimate for the known term g_ξ. From (2.22), (4.14), and (2.23),

$$\|g_\xi\|_\infty \leqslant C(\|\varphi_\xi\|_\infty^2 + \|\varphi_\xi\|_\infty + |\pi_\xi(G_\xi)| + |H(\xi)| + \|\partial_\xi n_\xi - \pi_\xi(\partial_\xi n_\xi)\, v_\xi)\|_\infty),$$

where we have used $\|\partial_\xi p_\xi\|_\infty \leqslant \text{const}\, \|\partial_\xi n_\xi\|_\infty$ and (2.14), and $\|\partial_\xi v_\xi\|_\infty \leqslant \text{const}$, which follows by (2.18) and (2.25). Recalling $G_\xi = -(f(n_\xi) + R_\xi[\varphi_\xi])$, (2.5), (2.15), (2.16), and (2.22) we have that $|\pi_\xi(G_\xi)| \leqslant C(e^{-2\alpha\xi} + \|\varphi_\xi\|_\infty)$. From (2.24), (2.25), and $\|\partial_\xi p_\xi\|_\infty \leqslant \text{const}$, we also have that $|H(\xi)| \leqslant C\, \|\varphi_\xi\|_\infty$. Then, from the first bound in (4.9), (2.27), and the previous estimates, we conclude that $\|g_\xi\|_\infty \leqslant Ce^{-\min\{2\alpha;\, \delta_1\}\, \xi}$. The proposition is thus proved. ∎

To conclude the proof of Theorem 2.8 we are left with the equation $P(\xi) = 0$, where $P(\xi) = \pi_\xi(f(n_\xi) + R_\xi[\varphi_\xi])$. We first observe that from (2.15), (2.16), Lemma 2.4, and Proposition 4.2, for any $\kappa > 0$ there are $\ell_8 \geqslant \ell_7$ such that $|P(\xi) - V(\xi)| \leqslant Ce^{-(2\alpha+\delta_2)\,\xi}$ for any $h < e^{-2\alpha\ell_8}$ and $\xi \in \Gamma_{\ell_8, \kappa}$ (recall $\delta_2 = \min\{\delta_0; \delta_1; 2\alpha\}$). We then have:

$$P(\ell) \leqslant -(\mu K - Ce^{-\delta_2 \ell})\, e^{-2\alpha\ell} + 2m_\beta\, \mu h,$$

$$P((2\alpha)^{-1}\, |\log h| + \kappa) \geqslant [\mu(2m_\beta - Ke^{-2\alpha\kappa}) - Ce^{-(2\alpha+\delta_2)\,\kappa} h^{(2\alpha)^{-1}\,\delta_2}]\, h.$$

Let $\kappa_0 > 0$ be such that $2m_\beta - Ke^{-2\alpha\kappa_0} \geqslant m_\beta$ and, for any $\kappa \geqslant \kappa_0$, let ℓ_9 so large that $\mu K - Ce^{-\delta_2 \ell_9} \geqslant \mu K/2$. Then, for any $\kappa \geqslant \kappa_0$ and $\ell \geqslant \ell_9$ there is $h_{\kappa,\ell}$ such that, for any $h < h_{\kappa,\ell}$, we have $P(\ell) < 0$ and $P((2\alpha)^{-1}\, |\log h| + \kappa) > 0$. Since the function $\xi \mapsto P(\xi)$ is easily seen to be continuous (actually differentiable) it follows it has at least one zero in $\Gamma_{\ell,\kappa}$ for any h small enough. In order to prove uniqueness it is sufficient to show the function is strictly monotone. Recalling $G_\xi \doteq -(f(n_\xi) + R_\xi[\varphi_\xi])$, using (4.13), and $v_\xi^* L_\xi = \lambda_\xi v_\xi^*$ we have:

$$P'(\xi) = \lambda_\xi \pi_\xi(\partial_\xi n_\xi) - \pi_\xi(K_\xi + Z_\xi J * \partial_\xi \varphi_\xi) + \partial_\xi \pi_\xi(f(n_\xi)) + \partial_\xi \pi_\xi(R_\xi[\varphi_\xi]).$$

From (2.22), (2.26), (4.14), (4.15), and Proposition 4.2, it follows that, for all $h < e^{-2\alpha\ell_9}$ and $\xi \in \Gamma_{\ell_9, \kappa}$,

$$|\pi_\xi(K_\xi + Z_\xi J * \partial_\xi \varphi_\xi)| + |\partial_\xi \pi_\xi(R_\xi[\varphi_\xi])| \leqslant Ce^{-2(\alpha+\delta_2)\,\xi}.$$

On the other hand, from (2.15), (2.16), and using (2.29), $|\partial_\xi \pi_\xi(f(n_\xi))| \leqslant Ce^{-(2\alpha+\delta_2)\,\xi}$. From (2.18) and (2.28) we conclude that

$$P'(\xi) \geqslant \frac{e^{-2\alpha\xi}}{c_1 \sqrt{\mu}} (1 - Ce^{-\delta_2 \xi}).$$

Then, for any $\kappa \geqslant \kappa_0$, there is $\ell_5 \geqslant \ell_9$ such that, for each $h \leqslant h_{\kappa, \ell_5}$, the equation $P(\xi) = 0$ has a unique solution on $\Gamma_{\ell_5, \kappa}$. Theorem 2.8 thus follows with $\bar{\xi}$ equal to such a solution, $\bar{\varphi} = \varphi_{\bar{\xi}}$, and $h_0 = h_{\kappa, \ell_5}$. We have only to check the limits (2.39). By our choice of ℓ_9 and since $\ell_5 \geqslant \ell_9$,

$$P(\xi) \leqslant -\frac{\mu K}{2} e^{-2\alpha\xi} + 2m_\beta \mu h \qquad \forall \xi \in \Gamma_{\ell_5, \kappa}, \quad \forall h \leqslant h_{\kappa, \ell_5},$$

which implies $\bar{\xi} \to +\infty$ as $h \downarrow 0$. On the other hand, since $|P(\xi) - V(\xi)| \leqslant Ce^{-(2\alpha+\delta_2)\xi}$,

$$|e^{2\alpha\bar{\xi}} V(\bar{\xi})| = |2m_\beta \mu h e^{2\alpha\bar{\xi}} - \mu K| \leqslant Ce^{-\delta_2 \bar{\xi}}.$$

Letting $h \downarrow 0$ in the above inequality we thus obtain the first limit in (2.39); the second one then follows by the first bound in (4.9). ∎

5. SPATIAL PROPERTIES OF THE BUMP

In this section we prove Proposition 2.9. The critical droplet is uniquely determined by its restriction to the semi-space $\mathbb{R}_+$, which solves

$$q(x) = \tanh\{\beta[(J_+ q)(x) + h]\}, \qquad x \in \mathbb{R}_+, \tag{5.1}$$

where, for any $u \in C(\mathbb{R}_+)$,

$$(J_\pm u)(x) \doteq \int_0^{+\infty} dy\, J_\pm(x, y)\, u(y), \qquad J_\pm(x, y) \doteq J(x-y) \pm J(x+y). \tag{5.2}$$

Observe that, since J is nonincreasing on $\mathbb{R}_+$, $J_\pm(x, y) \geqslant 0$ for all $x, y \geqslant 0$; moreover, since $\sup\{x \in \mathbb{R} : J(x) > 0\} = 1$, if $x > 1$ then $J_\pm(x, y) = J(x-y)$ for all $y \geqslant 0$. Let $\bar{m}_{\xi^*}$ be as in Theorem 2.1. Since $\beta(1 - m_\beta^2) < 1$, from (1.3) and (2.9) there are $\theta \in (0, 1)$, $h_1 \in (0, h_0]$, and a positive integer $\ell^* \in (1, \xi^* - 1)$ such that:

$$\beta(1 - \bar{m}_{\xi^*}(x)^2) \leqslant \theta \qquad \forall |x - \xi^*| \geqslant \ell^* - 1, \quad \forall h \in (0, h_1]. \tag{5.3}$$

On the other hand, recalling the definition (2.42), (2.9) implies

$$\lim_{h \downarrow 0} \|p - \beta(1 - \bar{m}_{\xi^*}^2)\|_\infty = 0. \tag{5.4}$$

From (5.3) and (5.4) it follows there are $\delta \in (\theta, 1)$ and $h_2 \in (0, h_1]$ such that, for $\ell^* \in (1, \xi^* - 1)$ as before,

$$p(x) \leqslant \delta \qquad \forall |x - \xi^*| \geqslant \ell^* - 1, \quad \forall h \in (0, h_2]. \tag{5.5}$$

Lemma 5.1. Let $h \in (0, h_2]$, h_2 and ℓ^* be as in (5.5). Then, for each $k \in [0, \xi^* - \ell^*]$ and $s \geqslant \xi^* + \ell^*$, we have:

$$q'(x) = \int_k^{k+1} dy\, H_k(x, y)\, q'(y) \qquad \forall x \in [0, k), \tag{5.6}$$

$$q'(x) = \int_{s-1}^{s} dy\, K_s(x, y)\, q'(y) \qquad \forall x \in (s, +\infty), \tag{5.7}$$

where $H_k(x, y)$, $x \in (0, k)$, and $K_s(x, y)$, $x > s$, are nonnegative continuous functions of y, strictly positive for some $y \in [k, k+1]$, $y \in [s-1, s]$, respectively.

Proof. We start with the proof of (5.6). We differentiate (5.1) at $x \in [0, k)$, obtaining (recall (5.2))

$$q'(x) = p(x) \int_0^k dy\, J_-(x, y)\, q'(y) + p(x) \int_k^{k+1} dy\, J_-(x, y)\, q'(y).$$

After N iteration we get

$$q'(x) = \int_k^{k+1} dy\, H_k^{(N)}(x, y)\, q'(y) + \int_0^k dy\, D_k^{(N)}(x, y)\, q'(y), \tag{5.8}$$

where

$$H_k^{(N)}(x, y) \doteq \sum_{n=1}^{N} D_k^{(n)}(x, y), \qquad D_k^{(1)}(x, y) \doteq p(x)\, J_-(x, y),$$

and, for $n > 1$, setting $x = y_0$ and $y = y_n$,

$$D_k^{(n)}(y_0, y_n) = \int_0^k dy_1 \cdots \int_0^k dy_{n-1} \prod_{i=1}^{n} p(y_{i-1})\, J_-(y_{i-1}, y_i).$$

The assumptions on J imply

$$0 \leqslant J_-(x, y) \leqslant J(x - y), \qquad J_-(0, y) \equiv 0 \qquad \forall x, y \in \mathbb{R}_+. \tag{5.9}$$

From (5.5), (5.9) and recalling $p(x) = \beta(1-q(x)^2)$, we get

$$0 \leqslant D_k^{(n)}(y_0, y_n) \leqslant \delta^{n-1} J^n(y_0, y_n). \tag{5.10}$$

Since $J^n(y_0, y_n)$ is a probability density and $\|q'\|_\infty < \infty$, the second integral in the r.h.s. of (5.8) vanishes as $N \to +\infty$ and we obtain (5.6) with

$$H_k(x, y) = \sum_{n=1}^{\infty} D_k^{(n)}(x, y), \tag{5.11}$$

and the series converges exponentially fast. Clearly $H_k(x, \cdot)$ is nonnegative and continuous. Moreover it is strictly positive for some $y \in [k, k+1]$ because, for all $x > 0$, $\sup\{|y-x| \in \mathbb{R}_+ : J_-(x, y) > 0\} > 0$.

The case $x > s$ can be treated in the same manner, getting

$$K_s(x, y) = \sum_{n=1}^{\infty} R_s^{(n)}(x, y), \qquad R_s^{(1)}(x, y) \doteq p(x)\, J(x-y), \tag{5.12}$$

where, for $n > 1$, setting $x = y_0$ and $y = y_n$,

$$R_s^{(n)}(y_0, y_n) = \int_s^{+\infty} dy_1 \cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^{n} p(y_{i-1})\, J(y_{i-1} - y_i). \tag{5.13}$$

In (5.12) we used that $J_-(u, v) = J(u-v)$ for $u > s > 1$ and $v \geqslant 0$. ∎

Proof of the Monotonicity Property. We first prove that there is $h_3 \in (0, h_2]$ such that

$$q'(x) < 0 \qquad \forall\, |x-\xi^*| \leqslant \ell^*, \quad \forall h \in (0, h_3]. \tag{5.14}$$

To prove (5.14) we differentiate (5.1) for $|x-\xi^*| \leqslant \ell^*$. Recalling $J_+(x, y) = J(x-y)$ when $x+y > 1$, we get,

$$q'(x) = p(x)(J * q)'(x) = p(x)(J' * (q - \bar{m}_{\xi^*}))(x) + p(x)(J * \bar{m}'_{\xi^*})(x). \tag{5.15}$$

Since $\bar{m}_{\xi^*}$ is strictly decreasing on $\mathbb{R}_+$, from (2.9) we get (5.14).

From (5.14) and Lemma 5.1 it follows $q'(x) < 0$ for all $x > 0$ and $h \in (0, h_3]$, thus getting the monotonicity property of the bump. ∎

We will prove Proposition 2.9 with $h^* = h_3$. We are thus left with the proof of (2.41). We follow the same strategy used in Section 3 of ref. 9. In fact large part of that strategy can be adapted to our context without modification. We first need a weaker result.

Lemma 5.2. There are $\eta > 0$ and $c > 0$ such that

$$|q'(x)| \leqslant c e^{-\eta(x-\xi_q)} \qquad \forall x \in \mathbb{R}_+, \quad \forall h \in (0, h^*], \tag{5.16}$$

where $\xi_q = \xi_q(h)$ is the (unique) positive zero of the bump.

Proof. By the definition (5.13), $R_s^{(n)}(x, y) = 0$ if $x > n+s$ and $y \in [s-1, s]$, and it satisfies a bound analogous to (5.10). Then, from (5.7), for any $x > s \geqslant \xi^* + \ell^*$, we have

$$|q'(x)| \leqslant \beta \, \|q'\|_\infty \sum_{n \geqslant x-s} \delta^{n-1} \leqslant \delta^{-1} \beta \, \|q'\|_\infty e^{-(x-s)\,|\log \delta|}. \tag{5.17}$$

Let ξ_q be the (unique) zero of $q(x)$ in $\mathbb{R}_+$. By (2.9) $\bar{m}_{\xi^*}(\xi_q) = \bar{m}(\xi^* - \xi_q)$ vanishes as $h \downarrow 0$, hence

$$\lim_{h \downarrow 0} \, [\xi_q(h) - \xi^*(h)] = 0. \tag{5.18}$$

In particular (5.18) implies there is $\ell < \infty$ such that

$$\xi_q + \ell \geqslant \xi^* + \ell^* \qquad \forall h \in [0, h^*]. \tag{5.19}$$

and (5.16) follows from (5.17) with $s = \xi_q + \ell$. ∎

From (5.7), we have, for each $s \geqslant \ell$,

$$q'(x + \xi_q) = \int_{s-1}^{s} dy \, G_s(x, y) \, q'(y + \xi_q), \qquad \forall h \in (0, h^*], \tag{5.20}$$

where, setting $p_{\xi_q}(x) \doteq p(x + \xi_q)$,

$$G_s(x, y) \doteq \sum_{n=1}^{\infty} \int_s^{+\infty} dy_1 \cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^{n} p_{\xi_q}(y_{i-1}) \, J(y_{i-1} - y_i). \tag{5.21}$$

We observe that $p_{\xi_q}(x)$ is a strictly decreasing function of x for $x > 0$,

$$p_{\xi_q}(x) > \inf_{x > 0} p_{\xi_q}(x) = p_\infty = \beta(1 - (m_{\beta, h}^-)^2) < 1 \tag{5.22}$$

and, by Lemma 5.2, there is $c' > 0$ such that

$$p_{\xi_q}(x) \leqslant p_\infty + c' e^{-\eta x}, \qquad \forall h \in [0, h^*]. \tag{5.23}$$

In Theorem 3.1 of ref. 9 the asymptotics of $\bar{m}'(x)$ follows from an analogous (to (5.20)) expression for $\bar{m}'(x)$, where $p_{\xi_q}(x)$ is replaced by $p_{\bar{m}}(x) \doteq \beta(1-\bar{m}(x)^2)$ in the definition of $G_s(x, y)$. The proof does not depend on the specific form of the function $p_{\bar{m}}(x)$, but only on the monotonicity property and the analogous of (5.22) and (5.23). Then a result as Theorem 3.1 of ref. 9 holds in our case. We conclude that there exist $M > 0$ and $\delta \in (0, \gamma)$, γ as in (2.40), such that

$$\lim_{x\to+\infty} e^{\gamma x} q'(x+\xi_q) = -M, \qquad \lim_{x\to+\infty} e^{\delta x}(e^{\gamma x} q'(x+\xi_q)+M) = 0. \tag{5.24}$$

As in ref. 9 the constant M is nonzero because of the monotonicity property of $q'(x)$. Moreover, since $0 < p_\infty < 1$ and (5.23) holds uniformly in h, the constant $M = M(h)$ appearing in (5.24) remains bounded away from 0 as $h \downarrow 0$.

Analogously we obtain (2.41) (with $A = M\gamma^{-1}$) from (5.24) by arguing exactly as in the proofs of Theorems 3.2 and 3.3 of ref. 9 where (2.7) follows as a corollary of Theorem 3.1 of ref. 9. We omit the details. ∎

6. THE INVARIANT MANIFOLD $\mathcal{W}$

In this section we prove Theorem 2.11, i.e., the existence of a one dimensional, invariant, expanding manifold $\mathcal{W}$ in $C^{\text{sym}}(\mathbb{R}; [-1, 1])$ consisting of two branches that originate from the bump q.

For $h \in (0, h^*]$ (h^* as in Proposition 2.9) let L, λ, v be as in Proposition 2.10. We next derive some properties of the evolution $S_t(q+u_0)$ starting from an initial datum $q+u_0$ with u_0 small. We set $u_t \doteq S_t(q+u_0)-q$. Since $S_t(q) = q$ and $f(q) = 0$ we have

$$\frac{du_t}{dt} = Lu_t + [f(q+u_t) - f(q) - Lu_t], \tag{6.1}$$

which implies

$$u_t = e^{Lt}u_0 + \int_0^t ds\, e^{L(t-s)}[f(q+u_s) - f(q) - Lu_s]. \tag{6.2}$$

Then by (2.5) and (2.47) (with $\zeta = 0$)

$$\|u_t - e^{Lt}u_0\|_\infty \leqslant C_2 \int_0^t ds\, e^{\lambda(t-s)}\, \|u_s\|_\infty^2, \qquad C_2 \doteq C_1 k_2. \tag{6.3}$$

Lemma 6.1. There is $N > 0$ such that if $u_0 \in L_\infty(\mathbb{R})$ satisfies

$$\sigma(u_0) \doteq \frac{1}{\lambda} \log \frac{1}{N \|u_0\|_\infty} > 0, \tag{6.4}$$

then, for all $t < \sigma(u_0)$,

$$\|u_t - e^{Lt} u_0\|_\infty \leqslant N(e^{\lambda t} \|u_0\|_\infty)^2 \tag{6.5}$$

and, with C_1 as in (2.47),

$$\|u_t\|_\infty \leqslant (1 + C_1)\, e^{\lambda t} \|u_0\|_\infty. \tag{6.6}$$

Proof. The lemma will follow with $N \doteq 4C_2 \lambda^{-1}$. We prove (6.5) by contradiction. Fix $\tau < \sigma(u_0)$ and define $\rho_\tau \doteq e^{\lambda \tau} \|u_0\|_\infty$. Let $T \leqslant \tau$ be the first time when the inequality (6.5) becomes an equality. Then, by (6.3) with $t = T$,

$$\begin{aligned} N(e^{\lambda T} \|u_0\|_\infty)^2 &\leqslant C_2 \int_0^T ds\, e^{\lambda(T-s)} [e^{\lambda s} \|u_0\|_\infty + N(e^{\lambda s} \|u_0\|_\infty)^2]^2 \\ &\leqslant C_2 (1 + N\rho_\tau)^2\, \lambda^{-1} (e^{\lambda T} \|u_0\|_\infty)^2 < N(e^{\lambda T} \|u_0\|_\infty)^2, \end{aligned} \tag{6.7}$$

where in the last inequality we used $N\rho_\tau < 1$. We have thus reached a contradiction and (6.5) is proved for all $t \leqslant \tau$. Hence, by (2.47),

$$\|u_t\|_\infty \leqslant C_1 e^{\lambda t} \|u_0\|_\infty + N(e^{\lambda t} \|u_0\|_\infty)^2 \leqslant (1 + C_1)\, e^{\lambda t} \|u_0\|_\infty \tag{6.8}$$

for all $t \leqslant \tau$, and Lemma 6.1 is proved. ∎

We use in the sequel the following notation. For v, N as in Proposition 2.10 and Lemma 6.1 we denote by ρ any positive number such that $N\rho \|v\|_\infty < 1$ and define

$$\psi_{\pm\varepsilon} \doteq q \pm \varepsilon v, \qquad \varepsilon \in [0, \rho], \qquad \tau(\rho, \varepsilon) \doteq \frac{1}{\lambda} \log \frac{\rho}{\varepsilon}. \tag{6.9}$$

We observe that $\pm \varepsilon v$, $\varepsilon \in [0, \rho]$ satisfy the hypothesis of Lemma 6.1 and that $\tau(\rho, \varepsilon) < \sigma(\pm \varepsilon v)$, $\sigma(\cdot)$ as in (6.4). Hence, for any $t \leqslant \tau(\rho, \varepsilon)$,

$$\|S_t(\psi_{\pm\varepsilon}) - (q \pm e^{\lambda t} \varepsilon v)\|_\infty \leqslant N(e^{\lambda t} \varepsilon \|v\|_\infty)^2, \tag{6.10}$$

$$\|S_t(\psi_{\pm\varepsilon}) - q\|_\infty \leqslant (1 + C_1)\, e^{\lambda t} \varepsilon \|v\|_\infty. \tag{6.11}$$

Theorem 6.2. For any $h \in (0, h^*]$ (h^* as in Proposition 2.9), there are $\rho > 0$ and $w_s^{\pm} \in C_0^{\text{sym}}(\mathbb{R})$, $s \leqslant 0$, such that, for any $s \leqslant 0$,

$$\lim_{\varepsilon \downarrow 0} \|S_{\tau(\rho,\,\varepsilon)+s}(\psi_{\pm\varepsilon}) - w_s^{\pm}\|_\infty = 0. \tag{6.12}$$

Moreover

$$\lim_{s \to -\infty} \|w_s^{\pm} - q\|_\infty = 0; \qquad S_t(w_s^{\pm}) = w_{s+t}^{\pm} \quad \text{if } s+t \leqslant 0. \tag{6.13}$$

A uniformity in $s \leqslant 0$ of the limit (6.12) is proved in Proposition 6.6 later to which we refer for a precise statement.

Proof of Theorem 6.2. We will next prove that if ρ is small enough then $\{S_{\tau(\rho,\,\varepsilon)}(\psi_{\pm\varepsilon})\colon \varepsilon \in (0, \rho]\}$ is a Cauchy sequence as $\varepsilon \downarrow 0$. Without loss of generality we restrict to the case with the plus sign. Observing that $\psi_\varepsilon = q + e^{\lambda\tau(\varepsilon,\,\varepsilon')}\varepsilon' v$ for any $0 < \varepsilon' < \varepsilon$, by (6.10),

$$\|S_{\tau(\varepsilon,\,\varepsilon')}(\psi_{\varepsilon'}) - \psi_\varepsilon\|_\infty \leqslant N\, \|v\|_\infty^2\, \varepsilon^2. \tag{6.14}$$

We thus need to compare $S_t(\psi_\varepsilon)$ and $S_t(\tilde{m})$, $t \leqslant \tau(\rho, \varepsilon)$, for all functions $\tilde{m}$ such that

$$\|\tilde{m} - \psi_\varepsilon\|_\infty \leqslant N\, \|v\|_\infty^2\, \varepsilon^2. \tag{6.15}$$

Let

$$\Delta_t \doteq S_t(\psi_\varepsilon) - S_t(\tilde{m}). \tag{6.16}$$

By (2.5) we have

$$\frac{d\Delta_t}{dt} = L_{S_t(\psi_\varepsilon)}\Delta_t + R_t^{(1)}, \qquad \|R_t^{(1)}\|_\infty \leqslant k_2\, \|\Delta_t\|_\infty^2. \tag{6.17}$$

Since $\|L_{m+u}\Delta - L_m\Delta\|_\infty \leqslant c'\, \|u\|_\infty\, \|\Delta\|_\infty$ with c' a suitable constant, by (6.11) there is C_3 so that

$$R_t^{(2)} \doteq L_{S_t(\psi_\varepsilon)}\Delta_t - L\Delta_t, \qquad \|R_t^{(2)}\|_\infty \leqslant C_3\rho\, \|\Delta_t\|_\infty \qquad \forall t \leqslant \tau(\rho, \varepsilon). \tag{6.18}$$

Thus

$$\Delta_t = e^{Lt}\Delta_0 + \int_0^t ds\, e^{L(t-s)}[R_s^{(2)} + R_s^{(1)}]. \tag{6.19}$$

Then by (2.47) and the bounds in (6.17) and (6.18), for any $t \leqslant \tau(\rho, \varepsilon)$,

$$\|\Delta_t\|_\infty \leqslant C_1 e^{\lambda t} \Delta_0 + C_1 \int_0^t ds\, e^{\lambda(t-s)} [C_3 \rho \, \|\Delta_s\|_\infty + k_2 \, \|\Delta_s\|_\infty^2].$$

By iteration and recalling (6.3), for $\lambda^* \doteq \lambda + C_2 C_3 \rho$, we have:

$$\|\Delta_t\|_\infty \leqslant C_1 e^{\lambda^* t} \Delta_0 + C_2 \int_0^t ds\, e^{\lambda^*(t-s)} \, \|\Delta_s\|_\infty^2. \tag{6.20}$$

Setting $W_t \doteq e^{-\lambda^* t} \, \|\Delta_t\|_\infty$ and using (6.15), from (6.20) we get, for all $t \leqslant \tau(\rho, \varepsilon)$,

$$W_t \leqslant C_1 N \, \|v\|_\infty^2 \, \varepsilon^2 + C_2 \int_0^t ds\, W_s^2, \tag{6.21}$$

which implies:

$$W_t \leqslant c\varepsilon^2 \sum_{n=0}^{\infty} (c\varepsilon^2 t)^n, \qquad c \doteq C_1 (1 \vee C_2) \, N \, \|v\|_\infty^2. \tag{6.22}$$

Since $\varepsilon\tau(\rho, \varepsilon) \to 0$ as $\varepsilon \downarrow 0$, we can choose $\varepsilon_1 \in (0, \rho]$ so that the series converges and $W_t \leqslant 2c\varepsilon^2$ for all $\varepsilon \in (0, \varepsilon_1]$ and $t \leqslant \tau(\rho, \varepsilon)$. We choose ρ small enough so that $C_2 C_3 \rho \leqslant \lambda/2$, i.e., $e^{\lambda^* \tau(\rho, \varepsilon)} \leqslant (\rho/\varepsilon)^{3/2}$. Then, recalling (6.9) and the definition of W_t we get, for $C_4 \doteq 2c\rho^{3/2}$,

$$\|\Delta_t\|_\infty \leqslant C_4 \sqrt{\varepsilon}, \qquad \forall \varepsilon \in (0, \varepsilon_1], \quad \forall t \leqslant \tau(\rho, \varepsilon). \tag{6.23}$$

By (6.14) and (6.23) we conclude that

$$\|S_{\tau(\rho, \varepsilon')}(\psi_{\varepsilon'}) - S_{\tau(\rho, \varepsilon)}(\psi_\varepsilon)\|_\infty \leqslant C_4 \sqrt{\varepsilon} \qquad \text{if} \quad 0 < \varepsilon' < \varepsilon \leqslant \varepsilon_1, \tag{6.24}$$

which shows $\{S_{\tau(\rho, \varepsilon)}(\psi_\varepsilon)\}$ is a Cauchy sequence as $\varepsilon \downarrow 0$ for all ρ small enough. The same argument shows that also $S_{\tau(\rho, \varepsilon)+s}(\psi_{\pm\varepsilon})$ is a Cauchy sequence for each $s \leqslant 0$. Then $S_{\tau(\rho, \varepsilon)+s}(\psi_{\pm\varepsilon})$ converges in sup norm as $\varepsilon \downarrow 0$ to a function $w_s^\pm$, hence (6.12). Moreover if $t+s \leqslant 0$, $t \geqslant 0$, then $S_t(S_{\tau(\rho, \varepsilon)+s}(\psi_\varepsilon)) = S_{\tau(\rho, \varepsilon)+s+t}(\psi_\varepsilon)$. By (2.2) for each $t \geqslant 0$, $S_t(m)$ depends continuously on m, thus $S_t(S_{\tau(\rho, \varepsilon)+s}(\psi_\varepsilon)) \to S_t(w_s^\pm)$ as $\varepsilon \downarrow 0$. On the other hand $S_{\tau(\rho, \varepsilon)+s+t}(\psi_\varepsilon) \to w_{s+t}^\pm$ as $\varepsilon \downarrow 0$, hence $S_t(w_s^\pm) = w_{s+t}^\pm$, proving the second relation in (6.13). Finally, from (6.11),

$$\|S_{\tau(\rho, \varepsilon)+s}(\psi_{\pm\varepsilon}) - q\|_\infty \leqslant C_5 e^{\lambda s}, \qquad C_5 \doteq (1 + C_1) \, \rho \, \|v\|_\infty, \tag{6.25}$$

from which, letting $\varepsilon \downarrow 0$,

$$\|w_s^{\pm} - q\|_\infty \leqslant C_5 e^{\lambda s}, \tag{6.26}$$

proving the first statement in (6.13), Theorem 6.2 is proved. ▮

Proof of Theorem 2.11. The manifold

$$\mathscr{W} \doteq \mathscr{W}^+ \cup \mathscr{W}^-, \qquad \mathscr{W}^{\pm} \doteq \{S_t(w_s^{\pm}) : s \leqslant 0 \leqslant t\} \tag{6.27}$$

and both its branches $\mathscr{W}^{\pm}$ are invariant under S_t which is invertible on $\mathscr{W}^{\pm}$. By (6.13) $\mathscr{W}^{\pm}$ originate at $s = -\infty$ from q. Recalling (6.9), from (6.10)

$$\|S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon}) - (q \pm e^{\lambda s} \rho v)\|_\infty \leqslant C_6 e^{2\lambda s}, \qquad C_6 \doteq N(\rho\, \|v\|_\infty)^2, \tag{6.28}$$

from which, letting $\varepsilon \downarrow 0$,

$$\|w_s^{\pm} - (q \pm e^{\lambda s} \rho v)\|_\infty \leqslant C_6 e^{2\lambda s}. \tag{6.29}$$

Next, by (6.1) and recalling that $f(q) = 0$,

$$\frac{dw_s^{\pm}}{ds} = L[w_s^{\pm} - (q \pm e^{\lambda s} \rho v)] \pm \lambda e^{\lambda s} \rho v + [f(w_s^{\pm}) - f(q) - L(w_s^{\pm} - q)]. \tag{6.30}$$

Denoting by $\|L\|_\infty$ the norm of the operator L (which is finite), by (2.5), (6.26), and (6.29) we have:

$$\left\| \frac{dw_s^{\pm}}{ds} \mp \lambda e^{\lambda s} \rho v \right\|_\infty \leqslant C_7 e^{2\lambda s}, \qquad C_7 \doteq \|L\|_\infty\, C_6 + k_2 C_5^2. \tag{6.31}$$

Recalling that $v(x) \approx \tilde{m}'(\xi^* - x)$ in the sense of (2.49), we set

$$s_0 : \rho e^{\lambda s_0} = \frac{1}{\sqrt{\mu}}, \qquad m_s^{\pm} \doteq w_{s+s_0}^{\pm}. \tag{6.32}$$

Then (6.31) implies

$$\left\| \frac{dm_s^{\pm}}{ds} \mp \lambda e^{\lambda s} \frac{1}{\sqrt{\mu}} v \right\|_\infty \leqslant C_7 e^{2\lambda s},$$

which gives (2.52).

The proofs of the monotonicity property of $m_s^{\pm}$ and of the bound (2.53) will be given in Proposition 6.5 later, Theorem 2.11 is then proved. ▮

We need the following properties of the flow S_t.

Theorem 6.3 (The Comparison Theorem[11]). Let $m, \tilde{m} \in L_\infty(\mathbb{R})$ be such that $m(x) \leqslant \tilde{m}(x)$ for all $x \in \mathbb{R}$. Then $S_t(m)(x) \leqslant S_t(\tilde{m})(x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

Lemma 6.4. Let $m \in L_\infty^{\mathrm{sym}}(\mathbb{R})$ be a nonincreasing function on $\mathbb{R}_+$. Then $S_t(m)$ has the same monotonicity property for all $t \in \mathbb{R}_+$.

Proof. The flow solution $S_t(m)$ solves the integral equation

$$S_t(m) = e^{-t}m + \int_0^t ds\, e^{-(t-s)} \tanh\{\beta[J * S_s(m) + h]\}.$$

Since J is smooth, the function $g_t(x) \doteq S_t(m)(x) - e^{-t}m(x)$ is differentiable. Further its spatial derivative $g_t'(x)$ is an antisymmetric function which satisfies, for any $x \in \mathbb{R}_+$,

$$\begin{aligned} g_t'(x) &= \int_0^t ds\, p_s(x)(J_- g_s')(x) + z_t(x), \\ p_s &\doteq \frac{\beta}{\cosh^2\{\beta[J * S_s(m) + h]\}}, \\ z_t &\doteq \int_0^t ds\, e^{-t} p_s(J_- * m'). \end{aligned} \tag{6.33}$$

To get (6.33) we used that, since g_s is differentiable, and since both g_s' and m' are odd functions,

$$\frac{d}{dx}(J * S_s(m)) = J * g_s' + e^{-s} J * m' = J_- g_s' + e^{-s} J_- m',$$

where for any function u on $\mathbb{R}_+$, $J_- u$, is defined in (5.2). By iteration of (6.33), calling $(t, x) = (s_0, x_0)$, we get

$$\begin{aligned} g_t'(x) = \sum_{n=1}^{\infty} \int_0^{s_0} ds_1 \cdots \int_0^{s_{n-1}} ds_n \\ \times \int_0^{+\infty} dx_1 \cdots \int_0^{+\infty} dx_n \prod_{k=1}^{n-1} p_s(x_k)\, J_-(x_k, x_{k+1})\, z_{s_n}(x_n). \end{aligned} \tag{6.34}$$

From the fact that $J_- \geqslant 0$ and $m'(x) \leqslant 0$ for $x \geqslant 0$ it follows that z_s is a nonpositive function on $\mathbb{R}_+$. Since $J_-(x, y)$ and $p_s(x)$ are nonnegative for

$x, y \geqslant 0$, we conclude from (6.34) that also g_t' is a non-positive function on $\mathbb{R}_+$. Then $S_t(m)$ is nonincreasing on $\mathbb{R}_+$ because sum of two functions with this property. The lemma is proved. ▮

Proposition 6.5. For any $s \in \mathbb{R}$, the symmetric functions $m_s^{\pm}$ are nonincreasing on $\mathbb{R}_+$ and (2.53) holds.

Proof. Since the difference between $m_s^{\pm}$ and $w_s^{\pm}$ is only a time shift, see (6.32), it is enough to prove the proposition for $w_s^{\pm}$.

We start with the monotonicity property. We use Theorem 6.2 and Lemma 6.4. Thus the first step is to show that for ε small the functions $\psi_{\pm\varepsilon} = q \pm \varepsilon v$ are nonincreasing on $\mathbb{R}_+$. To this purpose we first notice that, by definition of v,

$$v'(x) = -\frac{2\beta}{1+\lambda} q(x)\, q'(x)(J * v)(x) + \frac{p}{1+\lambda} (J' * v)(x). \tag{6.35}$$

By (2.46), for a suitable constant C_8,

$$\sup_{x>1} |e^{\gamma_v x} v'(x)| \leqslant C_8 \sup_{x>1} e^{\gamma_v x} v(x) < \infty. \tag{6.36}$$

Then from (2.41), since $\gamma_v > \gamma$, we get

$$\sup_{x>1} \left| \frac{v'(x)}{q'(x)} \right| < \infty. \tag{6.37}$$

For $x \in [0, 1]$, since $q'(0) = 0$, we need to show that $q''(0) \neq 0$. This is easily seen by noticing that since $q'(0) = 0$ and both J' and q' are anti-symmetric functions,

$$q''(0) = -2\beta(1 - q(0)^2) \int_0^1 dy\, J'(y)\, q'(y) < 0. \tag{6.38}$$

In the last inequality we used that, by our assumptions on the function J and Proposition 2.9, $J'(x)\, q'(x) > 0$ for $x \in (0, 1)$. From (6.37) and (6.38) we then get

$$\sup_{x \in \mathbb{R}_+} \left| \frac{v'(x)}{q'(x)} \right| < \infty. \tag{6.39}$$

Lemma 6.4 and (6.39) imply that for any $s \leqslant 0$ there is $\varepsilon_s \in (0, \rho]$ such that $\{S_{\tau(\rho, \varepsilon)+s}(\psi_{\pm\varepsilon}) : \varepsilon \in (0, \varepsilon_s]\}$ is a sequence of nonincreasing functions on $\mathbb{R}_+$.

Hence from (6.12) the same property holds for $w_s^\pm$, $s \leqslant 0$. Then the monotonicity property of $w_s^\pm$ for all $s \in \mathbb{R}$ follows from Lemma 6.4.

We are left with the bound (2.53). Since q solves (1.2) and it is strictly decreasing on $\mathbb{R}^+$, it follows that $m_{\beta,h}^- < q(x) < m_{\beta,h}^+$ for all $x \in \mathbb{R}$. We also recall that q satisfies (2.41). Since v is a positive function which satisfies (2.46) with $\gamma_v > \gamma$, we conclude that, for all ε small enough,

$$m_{\beta,h}^- \leqslant \psi_{-\varepsilon}(x) < q(x) < \psi_\varepsilon(x) < m_{\beta,h}^+. \tag{6.40}$$

Then (2.53) follows from Theorem 6.2 and the Comparison Theorem. ∎

We conclude this section by proving Proposition 6.6 below, which is a stronger version of Theorem 6.2, since we show that the curves $\{w_s^\pm\}$ are the limits, in sup norm, of the curves $S_{\tau(\rho,\delta)}\mathscr{C}_\delta$ where, for any $\delta > 0$, $\mathscr{C}_\delta \doteq \{\psi_\varepsilon\colon 0 < \varepsilon < \delta\}$.

Proposition 6.6. Let $\delta > 0$, $s \leqslant 0$, and $\delta(s) \doteq e^{\lambda s}\delta$. Then:

$$\lim_{\delta \downarrow 0} \sup_{s \leqslant 0} \|S_{\tau(\rho,\delta)}(\psi_{\pm\delta(s)}) - w_s^\pm\|_\infty = 0.$$

Proof. Without loss of generality we restrict to the case with the plus sign. We need to show that for any $\eta > 0$ there is $\delta_\eta > 0$ so that $\|S_{\tau(\rho,\delta)}(\psi_{\delta(s)}) - w_s^+\|_\infty \leqslant \eta$ for any $\delta < \delta_\eta$ and $s \leqslant 0$. We approximate w_s^+ by $S_t(\psi_\varepsilon)$ for suitable values of ε and t: given $s \leqslant 0$ let ε_0 be such that for $\varepsilon \in (0, \varepsilon_0]$

$$\|S_{\tau(\rho,\varepsilon)+s}(\psi_\varepsilon) - w_s^+\|_\infty \leqslant \frac{\eta}{2}. \tag{6.41}$$

For $\delta < \rho$ we have $S_{\tau(\rho,\varepsilon)+s}(\psi_\varepsilon) = S_{\tau(\rho,\delta)}(S_{\tau(\delta(s),\varepsilon)}(\psi_\varepsilon))$. By (6.10),

$$\begin{aligned}\|S_{\tau(\delta(s),\varepsilon)}(\psi_\varepsilon) - \psi_{\delta(s)}\|_\infty &= \|S_{\tau(\delta(s),\varepsilon)}(\psi_\varepsilon) - q - e^{\lambda\tau(\delta(s),\varepsilon)}\varepsilon v\|_\infty \\ &\leqslant N\,\|v\|_\infty^2\,\delta(s)^2.\end{aligned} \tag{6.42}$$

We define $D_t \doteq S_t(\psi_{\delta(s)}) - S_t(S_{\tau(\delta(s),\varepsilon)}(\psi_\varepsilon))$, so that

$$\|D_{\tau(\rho,\delta)}\|_\infty = \|S_{\tau(\rho,\varepsilon)+s}(\psi_\varepsilon) - S_{\tau(\rho,\delta)}(\psi_{\delta(s)})\|_\infty. \tag{6.43}$$

The analysis of D_t is identical to that of Δ_t in the proof of Theorem 6.2. In fact, by comparing (6.15) with (6.42), we see that D_t satisfies the conditions defining the function Δ_t in (6.16) when the parameter ε appearing in (6.15) and (6.16) is replaced by $\delta(s)$. Then the bound (6.23) applied to D_t

becomes: $\|D_t\|_\infty \leqslant C_4\sqrt{\delta(s)}$ for any $\delta \in (0, \varepsilon_1]$ and $t \leqslant \tau(\rho, \delta(s))$, which implies: $\|D_{\tau(\rho,\delta)}\|_\infty \leqslant C_4\sqrt{\delta}$ for all $\delta \in (0, \varepsilon_1]$. We then choose $\delta_\eta \in (0, \varepsilon_1]$ so small that $C_4\sqrt{\delta_\eta} \leqslant \eta/2$, and by (6.41) and (6.43) we have that, for all $\delta < \delta_\eta$,

$$\|S_{\tau(\rho,\delta)}(\psi_{\delta(s)}) - w_s^+\|_\infty \leqslant \|S_{\tau(\rho,\varepsilon)+s}(\psi_\varepsilon) - w_s^+\|_\infty + \|S_{\tau(\rho,\varepsilon)+s}(\psi_\varepsilon) - S_{\tau(\rho,\delta)}(\psi_{\delta(s)})\|_\infty$$
$$\leqslant \eta.$$

Proposition 6.6 is thus proved. ∎

7. GLOBAL STRUCTURE OF $\mathscr{W}$

In this section we prove Theorem 2.12. To this purpose we will define suitable functions $Q_a^+ \leqslant q \leqslant Q_a^-$, a a small parameter, which are close to q, see (7.9) later. We shall prove that the functions m_s^+ (resp. m_s^-) at a certain time s are above Q_a^- (resp. below Q_a^+). Then, by the Comparison Theorem it is enough to study the evolution of $Q_a^\pm$. Using the spectral properties of the linear operator L, we show that, for a time interval $T_a \sim |\log a|$, the evolution $S_{T_a}(Q_a^+)$ (resp. $S_{T_a}(Q_a^-)$) can be bounded from above (resp. below) by the same functions Q_a^+ (resp. Q_a^-) suitably translated in space, see Theorem 7.2 later. Thus, we can iterate the argument getting bounds at longer times which, combined with general properties of the flow S_t, lead to the desired result, Corollary 7.3 later, from which Theorem 2.12 will follow.

In the sequel we shall need a more refined *a priori* bound on the evolution around the critical droplet, which is the content of the following lemma.

Lemma 7.1. There is $K > 0$ such that if $u_t \doteq S_t(q + u_0) - q$, $u_0 \in L_\infty(\mathbb{R})$, then, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$|u_t(x) - e^{Lt}u_0(x)| \leqslant K \int_0^t ds\, e^{L(t-s)}(J * e^{Ls}u_0)^2\,(x) + K\mathscr{R}_t[u_\cdot], \tag{7.1}$$

where

$$\mathscr{R}_t[u_\cdot] = e^{\lambda t} \sup_{s \in [0,t]} \{\|u_s\|_\infty^3 + \|u_s\|_\infty\, \|u_s - e^{Ls}u_0\|_\infty + \|u_s - e^{Ls}u_0\|_\infty^2\}. \tag{7.2}$$

Proof. Recalling that u_t solves (6.2) and q solves (1.2) we have

$$f(q + u_s) - f(q) - Lu_s = \Phi(J * u_s)^2 + \frac{\beta^3}{3!}\tanh'''(\theta_s)(J * u_s)^3, \tag{7.3}$$

where $\Phi(x) \doteq -\beta^2 q(x)(1-q(x)^2)$ and θ_s is a number in the interval with end-points $\beta[J * q+h]$ and $\beta[J * (q+u_s)+h]$. Then we rewrite (6.2) as:

$$u_t = e^{Lt}u_0 + \int_0^t ds\, e^{L(t-s)}[\Phi(J * e^{Ls}u_0)^2 + R[u_s]], \tag{7.4}$$

where, using (7.3),

$$R[u_s] = \Phi[(J * u_s)^2 - (J * e^{Ls}u_0)^2] + \frac{\beta^3}{3!}\tanh'''(\theta_s)(J * u_s)^3. \tag{7.5}$$

Since Φ is a bounded function on $\mathbb{R}$, the first integral on the r.h.s. of (7.4) is bounded by the first term on the r.h.s. of (7.1) with $K = \|\Phi\|_\infty$. We next rewrite the square bracket on the r.h.s. of (7.5) as

$$(J * u_s)^2 - (J * e^{Ls}u_0)^2 = [J * (u_s - e^{Ls}u_0)]^2 + 2(J * e^{Ls}u_0)[J * (u_s - e^{Ls}u_0)].$$

Then, using $\tanh'''$ is bounded and J has compact support, from (2.47) we have, for any K large enough,

$$\left\| \int_0^t ds\, e^{L(t-s)} R[u_s] \right\|_\infty \leqslant K \mathscr{R}_t[u_\cdot].$$

The lemma is proved. ∎

Warning. For the rest of the section we shall denote by C a generic constant whose numerical value may change from line to line.

Let γ and λ be as in (2.40) and (2.44) respectively. We fix δ and R_0 such that

$$0 < \delta < \tfrac{1}{8}, \qquad \tfrac{3}{2} < \gamma R_0 < 2 - 4\delta, \tag{7.6}$$

and we set, for any $a \in (0, 1]$,

$$T_a \doteq \frac{\delta}{\lambda}\,|\log a|, \qquad R_a \doteq R_0\,|\log a|, \qquad \Delta_a \doteq a^{1-\delta/2}. \tag{7.7}$$

Recalling (2.44), (2.45), and (2.48), there exists an $\bar{h} \in (0, h^*]$ such that

$$(\gamma_v - \gamma)\, R_0 \leqslant \frac{\delta}{4} \qquad \text{and} \qquad \delta\lambda^{-1}\omega > 3 \qquad \forall h \in [0, \bar{h}]. \tag{7.8}$$

We define the symmetric functions

$$Q_a^{\pm}(x) \doteq q_a^{\pm}(x)\, \mathbf{1}_{|x| \leqslant R_a} + [m_{\beta,h}^{-} \pm a^{3/2}]\, \mathbf{1}_{|x|>R_a}, \tag{7.9}$$

$$q_a^{+}(x) \doteq q(|x|+a), \qquad q_a^{-}(x) \doteq q(0)\, \mathbf{1}_{|x| \leqslant a} + q(|x|-a)\, \mathbf{1}_{|x|>a}. \tag{7.10}$$

Theorem 7.2. Let $h \in [0, \bar{h}]$ with $\bar{h}$ as in (7.8). Then there is $a_0 \in (0, 1]$ such that, for any $a \in (0, a_0]$,

$$S_{T_a}(Q_a^{+})(x) \leqslant Q_a^{+}(x+\varDelta_a), \qquad S_{T_a}(Q_a^{-})(x) \geqslant Q_a^{-}(x-\varDelta_a), \tag{7.11}$$

with T_a and $\varDelta_a$ as in (7.7).

Proof. From Proposition 2.9 there is a constant c such that $|q''(x)| \leqslant c\, |q'(x)|$ for any $|x| \geqslant 1$. Then, by expanding to second order $q_a^{\pm}(x)$ around $q(x)$ for $|x| \geqslant 1$, and using $q_a^{+}(x) \leqslant q(x) \leqslant q_a^{-}(x)$ for all $x \in \mathbb{R}$, we have for any a small enough,

$$q_a^{+}(x) \leqslant q(x) + \frac{a}{2}\, q'(|x|)\, \mathbf{1}_{|x| \geqslant 1}, \qquad q_a^{-}(x) \geqslant q(x) - \frac{a}{2}\, q'(|x|)\, \mathbf{1}_{|x| \geqslant 1}. \tag{7.12}$$

Observing $q'(|x|) = -|q'(x)|$ for any $x \in \mathbb{R}$, if we define

$$\varphi(x) \doteq \tfrac{1}{2}\, |q'(x)|\, \mathbf{1}_{|x| \geqslant 1}, \tag{7.13}$$

from (7.9) and (7.12) we obtain:

$$Q_a^{+}(x) \leqslant q(x) - a\varphi(x) + [m_{\beta,h}^{-} - q(x) + a^{3/2} + a\varphi(x)]\, \mathbf{1}_{|x|>R_a}, \tag{7.14}$$

$$Q_a^{-}(x) \geqslant q(x) + a\varphi(x) + [m_{\beta,h}^{-} - q(x) - a^{3/2} - a\varphi(x)]\, \mathbf{1}_{|x|>R_a}. \tag{7.15}$$

Moreover, from (2.41) and (7.6), for any a small enough,

$$|m_{\beta,h}^{-} - q(x)| + a\, |\varphi(x)| \leqslant \tfrac{1}{2}\, a^{3/2} \qquad \forall |x| > R_a,$$

so that, if we define

$$U_0^{\pm}(x) \doteq \mp a\varphi(x) \pm \tfrac{3}{2}\, a^{3/2}\, \mathbf{1}_{|x|>R_a}, \tag{7.16}$$

from (7.14) and (7.15) we get, for any a small enough,

$$Q_a^{+}(x) \leqslant q(x) + U_0^{+}(x), \qquad Q_a^{-}(x) \geqslant q(x) + U_0^{-}(x). \tag{7.17}$$

We shall now obtain good bounds on $S_{T_a}(q+U_0^\pm)$. We can apply Lemma 7.1 to $U_t^\pm \doteq S_t(q+U_0^\pm)-q$, getting:

$$|U_t^\pm - e^{Lt}U_0^\pm| \leqslant K \int_0^t ds\, e^{L(t-s)}(J * e^{Ls}U_0^\pm)^2 + K\mathscr{R}_t[U_\cdot^\pm]. \tag{7.18}$$

We will use (7.18) to estimate $U_{T_a}^\pm$, by analyzing separately the various terms.

Estimate on $e^{LT_a}U_0^\pm$. Since $e^{\lambda T_a} = a^{-\delta}$, see (7.7), we have

$$e^{LT_a}U_0^\pm = \mp a^{1-\delta}\pi(\varphi)\, v \mp a e^{LT_a}[\varphi - \pi(\varphi)\, v] \pm \tfrac{1}{2}\, a^{3/2} e^{LT_a} \mathbf{1}_{|x|>R_a}, \tag{7.19}$$

where, recalling the definition of $\pi(\cdot)$ and (7.13),

$$\pi(\varphi) = \int_1^\infty dx\, \frac{v(x)}{p(x)}\, |q'(x)| > 0. \tag{7.20}$$

From the spectral gap property (2.48) and (7.8),

$$\|e^{LT_a}[\varphi - \pi(\varphi)\, v]\|_\infty \leqslant e^{-\omega T_a}\, \|\varphi - \pi(\varphi)\, v\|_\infty \leqslant Ca^3. \tag{7.21}$$

Analogously we estimate:

$$\begin{aligned} e^{LT_a}\mathbf{1}_{|x|>R_a} &= a^{-\delta}\pi(\mathbf{1}_{|x|>R_a})\, v + e^{LT_a}[\mathbf{1}_{|x|>R_a} - \pi(\mathbf{1}_{|x|>R_a})\, v] \\ &\leqslant Ca^{3/2-\delta}, \end{aligned} \tag{7.22}$$

where we used $\pi(\mathbf{1}_{|x|>R_a}) \leqslant Ca^{\gamma_v R_0}$ with $\gamma_v R_0 > \gamma R_0 > 3/2$. From (7.19), (7.21), and (7.22) we obtain:

$$|e^{LT_a}U_0^\pm \pm a^{1-\delta}\pi(\varphi)\, v| \leqslant Ca^{3-\delta}. \tag{7.23}$$

Estimate on $\int_0^{T_a} ds\, e^{L(T_a-s)}(J * e^{Ls}U_0^\pm)^2$. Using (2.47) with $\zeta = 0$ and (7.22), we get, for any a small enough,

$$\begin{aligned} &\int_0^{T_a} ds\, e^{L(T_a-s)}(J * e^{Ls}U_0^\pm)^2 \\ &\quad \leqslant Ca^2 \int_0^{T_a} ds\, e^{L(T_a-s)}[(J * e^{Ls}\varphi)^2 \\ &\qquad + a(J * e^{Ls}\mathbf{1}_{|x|\geqslant R_a})^2 + \sqrt{a}(J * e^{Ls}\varphi)(J * e^{Ls}\mathbf{1}_{|x|\geqslant R_a})] \\ &\quad \leqslant Ca^{3-2\delta} + Ca^2 \int_0^{T_a} ds\, e^{L(T_a-s)}(J * e^{Ls}\varphi)[(J * e^{Ls}\varphi) + a^{2-\delta}]. \end{aligned} \tag{7.24}$$

Now, recalling the definitions (7.13) and (2.43), from the asymptotics (2.41) it follows that $\varphi \in X_\gamma$. Since $\gamma < \gamma_v$ we can use (2.47) with $\zeta = \gamma$. Hence, since J has compact support, $|(J * e^{Ls}\varphi)(x)| \leqslant Ce^{\lambda s - \gamma |x|}$. Therefore, by applying again (2.47) with $\zeta = \gamma$,

$$\int_0^{T_a} ds\, e^{L(T_a - s)}(J * e^{Ls}\varphi)^2 \leqslant Ca^{-2\delta}e^{-\gamma |x|},$$

so that from (7.24), for all a small enough,

$$\int_0^{T_a} ds\, e^{L(t-s)}(J * e^{Ls}U_0^\pm)^2 \leqslant C(a^{2-2\delta}e^{-\gamma |x|} + a^{3-2\delta}). \tag{7.25}$$

Estimate on $\mathscr{R}_t[U_\cdot^\pm]$. We use Lemma 6.1 to obtain *a priori* bounds. Since $\|U_0^\pm\|_\infty \leqslant Ca$, comparing the definitions (6.4) and (7.7) and using $\delta < 1$ we conclude that for all a small enough $\sigma(U_0^\pm) > T_a$. Therefore from (6.5) and (6.6)

$$\|U_t^\pm\|_\infty \leqslant (1 + C_1)\, a^{1-\delta}, \qquad \|U_t^\pm - e^{Lt}U_0^\pm\|_\infty \leqslant Na^{2-2\delta} \qquad \forall t \leqslant T_a. \tag{7.26}$$

Recalling (7.2), from (7.26) we get

$$\mathscr{R}_{T_a}[U_\cdot^\pm] \leqslant Ca^{3-4\delta}. \tag{7.27}$$

Collecting (7.18), (7.23), (7.25), and (7.27), we conclude that, for any a small enough,

$$|U_{T_a}^\pm(x) \pm a^{1-\delta}\pi(\varphi)\, v(x)| \leqslant C(a^{2-2\delta}e^{-\gamma |x|} + a^{3-4\delta}). \tag{7.28}$$

Therefore, from the Comparison Theorem and (7.17), recalling $U_t^\pm = S_t(q + U_0^\pm) - q$, we finally get

$$S_{T_a}(Q_a^+)(x) \leqslant q(x) - C[a^{1-\delta}v(x) - a^{2-2\delta}e^{-\gamma |x|} - a^{3-4\delta}], \tag{7.29}$$

$$S_{T_a}(Q_a^-)(x) \geqslant q(x) + C[a^{1-\delta}v(x) - a^{2-2\delta}e^{-\gamma |x|} - a^{3-4\delta}]. \tag{7.30}$$

Next we shall find good bounds on $Q_a^\pm(x \pm \varDelta_a)$. We first observe that, since $\varDelta_a \leqslant 1$, $|x| \leqslant R_a - 1$ implies $|x + \varDelta_a| \leqslant R_a$ while $|x| > R_a + 1$ implies $|x + \varDelta_a| > R_a$. Moreover, from (2.41), (7.6), and (7.10), $|q_a^\pm(x \pm \varDelta_a) - m_{\beta,h}^-|$ $\leqslant a^{3/2}$ if $R_a - 1 < |x| \leqslant R_a + 1$ and a is small enough. Hence:

$$Q_a^+(x + \varDelta_a) \geqslant q_a^+(x + \varDelta_a)\, \mathbf{1}_{|x| \leqslant R_a + 1} + [m_{\beta,h}^- + a^{3/2}]\, \mathbf{1}_{|x| > R_a + 1},$$

$$Q_a^-(x - \varDelta_a) \leqslant q_a^-(x - \varDelta_a)\, \mathbf{1}_{|x| \leqslant R_a + 1} + [m_{\beta,h}^- - a^{3/2}]\, \mathbf{1}_{|x| > R_a + 1}.$$

Now we notice $q_a^+(x+\Delta_a) \geqslant q_{a+\Delta_a}^+(x)$ and $q_a^-(x-\Delta_a) \leqslant q_{a+\Delta_a}^-(x)$ for all $x \in \mathbb{R}$. Moreover, since $|q''(x)| \leqslant c\,|q'(x)|$ for $|x| \geqslant 1$, by expanding to the second order for $|x|>1$ and to the first one for $|x| \leqslant 1$, we get, if a is small enough,

$$q_{a+\Delta_a}^+(x) \geqslant q(x)-(a+\Delta_a)\,\psi(x), \qquad q_{a+\Delta_a}^-(x) \leqslant q(x)+(a+\Delta_a)\,\psi(x), \tag{7.31}$$

where

$$\psi(x) \doteq 2[|q'(x)|+\|q'\|_\infty\, \mathbf{1}_{|x| \leqslant 1}], \tag{7.32}$$

hence

$$Q_a^+(x+\Delta_a) \geqslant [q(x)-(a+\Delta_a)\,\psi(x)]\,\mathbf{1}_{|x| \leqslant R_a+1}+[m_{\beta,h}^- + a^{3/2}]\,\mathbf{1}_{|x|>R_a+1}, \tag{7.33}$$

$$Q_a^-(x-\Delta_a) \leqslant [q(x)+(a+\Delta_a)\,\psi(x)]\,\mathbf{1}_{|x| \leqslant R_a+1}+[m_{\beta,h}^- - a^{3/2}]\,\mathbf{1}_{|x|>R_a+1}. \tag{7.34}$$

We can now conclude the proof of the theorem. We consider first the case $|x| \leqslant R_a+1$. Since v is strictly positive and obeys the asymptotics (2.46), and q satisfies (2.41), from (7.8) and (7.32) we have

$$v(x) \geqslant C a^{\delta/4} \psi(x) \qquad \forall |x| \leqslant R_a+1. \tag{7.35}$$

On the other hand, using (7.6), $a^{2-2\delta} e^{-\gamma|x|}+a^{3-4\delta} \leqslant Ca\psi(x)$ for all $|x| \leqslant R_a+1$. Therefore, for any a small enough,

$$a^{1-\delta} v(x)-a^{2-2\delta} e^{-\gamma|x|}-a^{3-4\delta} \geqslant C a^{1-3\delta/4} \psi(x) \qquad \forall |x| \leqslant R_a+1. \tag{7.36}$$

Since $(a+\Delta_a)\,a^{-1+3\delta/4}$ vanishes as $a \downarrow 0$, (7.11) for $|x| \leqslant R_a+1$ follow from (7.29), (7.30), (7.33), (7.34), and (7.36).

Finally we consider the case $|x|>R_a+1$. Using $\gamma_v R_0 > \gamma R_0$ and (7.6), from (7.29) and (7.30) we get

$$\lim_{a \downarrow 0} a^{-3/2} \sup_{|x|>R_a+1} |S_{T_a}(Q_a^\pm)(x)-m_{\beta,h}^-| = 0. \tag{7.37}$$

Then (7.11) for $|x|>R_a+1$ and a small enough follow from (7.33), (7.34), and (7.37). ∎

Corollary 7.3. In the same hypothesis of Theorem 7.2, there is $a_1 \in (0, a_0]$ such that, for any $a \in (0, a_1]$,

$$\lim_{t \to +\infty} \|S_t(Q_a^+) - m_{\beta,h}^-\|_\infty = 0, \tag{7.38}$$

$$\lim_{t \to +\infty} S_t(Q_a^-)(x) = m_{\beta,h}^+ \qquad \forall x \in \mathbb{R}. \tag{7.39}$$

To prove the above corollary we need the following Barrier Lemma.

Lemma 7.4 (The Barrier Lemma[11]). There are V and C^* positive so that if $m, \tilde{m} \in L_\infty(\mathbb{R}; [-1, 1])$ and, for some $x_0 \in \mathbb{R}$ and $T > 0$, $m(x) = \tilde{m}(x)$ for all $|x - x_0| \leqslant VT$, then:

$$|S_t(m)(x_0) - S_t(\tilde{m})(x_0)| \leqslant C^* e^{-T}.$$

Proof of Corollary 7.3. We first prove (7.38). By (7.11) and the Comparison Theorem, $S_{nT_a}(Q_a^+)(x) \leqslant Q_a^+(x + n\Delta_a)$ for any integer n and $x \in \mathbb{R}$. From (7.9) the function on the r.h.s. of the above inequality is identically equal to $m_{\beta,h}^- + a^{3/2}$ for all $x > R_a - n\Delta_a$. On the other hand $S_{nT_a}(Q_a^+)$ is a symmetric function for all integer n, then $S_{nT_a}(Q_a^+)(x) \leqslant m_{\beta,h}^- + a^{3/2}$ for any $x \in \mathbb{R}$ and $n > R_a / \Delta_a$. Using again the Comparison Theorem and recalling $m_{\beta,h}^- \leqslant Q_a^+$, we conclude that:

$$m_{\beta,h}^- \leqslant S_t(Q_a^+) \leqslant S_t(m_{\beta,h}^- + a^{3/2}) \qquad \forall t > \left(1 + \frac{R_a}{\Delta_a}\right) T_a. \tag{7.40}$$

We now observe that $S_t(m_{\beta,h}^- + a^{3/2})$ solves the homogeneous equation:

$$\frac{d\rho(t)}{dt} = -\rho(t) + \tanh\{\beta[\rho(t) + h]\}, \tag{7.41}$$

with initial datum $m_{\beta,h}^- + a^{3/2}$. Since the interval $(-1, m_{\beta,h}^0)$ is a basin of attraction of the stationary solution $m_{\beta,h}^-$, for any a small enough,

$$\lim_{t \to +\infty} S_t(m_{\beta,h}^- + a^{3/2}) = m_{\beta,h}^-. \tag{7.42}$$

From (7.40) and (7.42) we get (7.38).

We shall next prove (7.39). We need to show that for any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ there is T_{ε, x_0} so that

$$|S_t(Q_a^-)(x_0) - m_{\beta,h}^+| < \varepsilon \qquad \forall t > T_{\varepsilon, x_0}. \tag{7.43}$$

By (7.11) and the Comparison Theorem, $S_{nT_a}(Q_a^-)(x) \geqslant Q_a^-(x - n\Delta_a)$ for any integer n and $x \in \mathbb{R}$. Recalling the definition (7.9), $S_{nT_a}(Q_a^-)(x) \geqslant q(x - n\Delta_a - a)$ for any $x > n\Delta_a$. Since Q_a^- is symmetric and nonincreasing for $x > 0$, from Lemma 6.4, we get $S_{nT_a}(Q_a^-)(x) \geqslant q(0)$ for all $x \in [0, n\Delta_a]$. Hence:

$$S_{nT_a}(Q_a^-)(x) \geqslant q(0)\, \mathbf{1}_{|x| \leqslant n\Delta_a} + q(|x| - n\Delta_a)\, \mathbf{1}_{|x| \geqslant n\Delta_a}.$$

We conclude that for any $R > 0$ we can find a time T_R so that

$$S_t(Q_a^-)(x) \geqslant q(0) \qquad \forall |x| \leqslant R, \quad \forall t > T_R. \tag{7.44}$$

We can now prove (7.43). Given any $\varepsilon > 0$ we choose T_ε so large that $C^* e^{-T_\varepsilon} < \varepsilon/2$ and $R \geqslant |x_0| + VT_\varepsilon$, C^*, V as in the Barrier Lemma 7.4. Hence from the Comparison Theorem, (7.44), and the Barrier Lemma it follows that:

$$S_t(Q_a^-)(x_0) > S_t(q(0)) - \frac{\varepsilon}{2} \qquad \forall t > T_R + T_\varepsilon. \tag{7.45}$$

On the other hand since $q(0)$ belongs to the basin of attraction of $m_{\beta,h}^+$ w.r.t. the dynamics (7.41) (in fact $m_{\beta,h}^0 < 0 < q(0) < m_{\beta,h}^+$), there is $\bar{T}$ such that:

$$|S_t(q(0)) - m_{\beta,h}^+| < \frac{\varepsilon}{2} \qquad \forall t > \bar{T}. \tag{7.46}$$

Recalling $Q_a^- \leqslant m_{\beta,h}^+$, from the Comparison Theorem, (7.45), and (7.46) we finally get

$$m_{\beta,h}^+ - \varepsilon < S_t(Q_a^-) \leqslant m_{\beta,h}^+ \qquad \forall t > \bar{T} \vee (T_\varepsilon + T_R),$$

which implies (7.43) with $T_{\varepsilon, x_0} = \bar{T} \vee (T_\varepsilon + T_R)$. ∎

Proof of Theorem 2.12. Let $w_s^\pm$, $s \leqslant 0$, be as in Theorem 6.2. We will next prove that for any $a > 0$ small enough there is $s_a < 0$ such that

$$w_{s_a}^-(x) \leqslant Q_a^+(x) \qquad \text{and} \qquad Q_a^-(x) \leqslant w_{s_a}^+(x) \qquad \forall x \in \mathbb{R}. \tag{7.47}$$

Theorem 2.12 will then follows from the Comparison Theorem, Corollary 7.3, (2.53), and (7.47) (recall that the relation between $w_s^\pm$ and $m_s^\pm$ is only a time shift, see (6.32)).

To prove (7.47) we need a more accurate estimate on the difference $w_s^\pm - (q \pm e^{\lambda s} \rho v)$. This is the content of Proposition 7.5 later.

Proposition 7.5. Let $w_s^\pm$ be as in Theorem 6.2 and γ as in (2.40). Then there is a constant $\bar{C}$ so that, for all $x \in \mathbb{R}$ and $s \leqslant 0$,

$$|w_s^\pm(x) - q \mp e^{\lambda s} \rho v(x)| \leqslant \bar{C}(e^{2\lambda s - \gamma|x|} + e^{3\lambda s}). \tag{7.48}$$

Proof. We apply Lemma 7.1 with $u_0 = \pm \varepsilon v$, getting

$$|S_t(\psi_{\pm\varepsilon}) - q \mp e^{\lambda s} \varepsilon v(x)| \leqslant K\varepsilon^2 \int_0^t dt'\, e^{2\lambda t'} e^{L(t-t')} (J * v)^2 + K \mathscr{R}_t[S_\cdot(\psi_{\pm\varepsilon}) - q]. \tag{7.49}$$

We bound $(J * v)^2$ by $\|v\|_\infty\, J * v$; we next observe that since J has compact support and v satisfies (2.46) with $\gamma_v > \gamma$, hence $J * v \in X_\gamma$, see (2.43). Then, by applying (2.47) with $\zeta = \gamma$,

$$K\varepsilon^2 \int_0^t dt'\, e^{2\lambda t'} e^{L(t-t')} (J * v)^2 \leqslant KC_1\, \|J * v\|_\gamma\, \|v\|_\infty\, \lambda^{-1} \varepsilon^2 e^{2\lambda t - \gamma|x|}. \tag{7.50}$$

We now observe that for $t = \tau(\rho, \varepsilon) + s$, $s \leqslant 0$, the r.h.s. of (7.50) is bounded, uniformly as $\varepsilon \downarrow 0$, by const $e^{2\lambda s - \gamma|x|}$. Analogously, from (6.25), (6.28), and (7.2), we get that $K\mathscr{R}_{\tau(\rho,\varepsilon)+s}[S_\cdot(\psi_{\pm\varepsilon}) - q]$ is bounded by const $e^{3\lambda s}$. The proposition is proved. ∎

Proof of (7.47). We first observe that, arguing as in getting (7.33) and (7.34), for all a small enough we have:

$$Q_a^+(x) \geqslant [q(x) - a\psi(x)]\, \mathbf{1}_{|x| \leqslant R_a} + [m_{\beta,h}^- + a^{3/2}]\, \mathbf{1}_{|x| > R_a}, \tag{7.51}$$

$$Q_a^-(x) \leqslant [q(x) + a\psi(x)]\, \mathbf{1}_{|x| \leqslant R_a} + [m_{\beta,h}^- - a^{3/2}]\, \mathbf{1}_{|x| > R_a}, \tag{7.52}$$

where ψ is defined in (7.32). We then set

$$s_a \doteq \frac{r}{\lambda} \log a \qquad \text{with} \quad \frac{1}{2} < r < 1 - \frac{\delta}{4} \text{ and } \delta \text{ as in (7.6).} \tag{7.53}$$

Recalling (2.46) and that $\gamma_v R_0 > \gamma R > 3/2$, from (7.48) it follows

$$\lim_{a \downarrow 0} a^{-3/2} \sup_{|x| > R_a} |w_{s_a}^\pm(x) - m_{\beta,h}^-| = 0. \tag{7.54}$$

On the other hand, by using (7.35), we also have, if a is small enough,

$$e^{\lambda s_a} \rho v(x) > a\psi(x) \qquad \forall x \in \mathbb{R}. \tag{7.55}$$

Then (7.47) follows from (7.48), (7.51)–(7.55). Theorem 2.12 is proved. ∎

8. THE QUASI-INVARIANT MANIFOLD AND THE SPECTRAL ESTIMATES

In this section we prove Theorem 2.2, Proposition 2.3, Lemma 2.4, and Theorem 2.5. Most of the statements follow from the results proved in refs. 9 and 10, as we are going to explain.

The first step is the definition of the function n_ξ which characterizes the quasi-invariant manifold $\mathcal{M}_{\ell,\kappa} = \{n_\xi : \xi \in \Gamma_{\ell,\kappa}\}$. As we already mentioned in the Introduction, the existence of the critical droplet has been proved in ref. 10 by using the Newton method: n_ξ will be a slight modification of the starting point of the Newton map.

Recalling the properties of the instanton stated in (2.7), we consider the following symmetric function:

$$m^0_\xi(x) \doteq \bar m_\xi(x) - a e^{-\alpha(\xi+|x|)}, \qquad \bar m_\xi(x) = \bar m(\xi - |x|). \tag{8.1}$$

The operator L_m (see (2.3)) for m in a neighbor of m^0_ξ has been studied in details in Theorems 2.1, 2.3, and 2.4 of ref. 9, see also Theorem 3.2 of ref. 10, and for the reader convenience we now recall these results.

We first define the neighbor of m^0_ξ as the set $G(c^*, \xi, \delta^*)$, $\xi > 1$, $c^*, \delta^* > 0$ of all functions $m \in C^{\mathrm{sym}}(\mathbb{R}, [-1, 1])$ such that

$$|m(x) - m^0_\xi(x)| \leqslant c^* \begin{cases} e^{-2\alpha\xi} e^{\alpha(\xi - x)} & \text{for} \quad 0 \leqslant x \leqslant \xi \\ e^{-2\alpha\xi} & \text{for} \quad \xi < x \end{cases} \tag{8.2}$$

and

$$-\int_{|x-\xi| \leqslant \sqrt{\xi}} dx [m(x) - m^0_\xi(x)] \, \bar m'(\xi - x)^2 \, \bar m(\xi - x) \geqslant -c^* e^{-(2\alpha + \delta^*)\xi}. \tag{8.3}$$

Spectral Analysis in $G(c^*, \xi, \delta^*)$. Some of the results that we need are true for ξ sufficiently large, so we assume this condition even if it is not always necessary. In ref. 9 the results are stated in the interval $[0, \ell]$ with Neumann boundary conditions, but since the estimates are uniform in ℓ they include the case $\ell = \infty$ treated here.

For all $m \in G(c^*, \xi, \delta^*)$ the operator L_m has a strictly positive eigenvalue λ_m with strictly positive right and left eigenfunctions denoted, respectively, by v^*_m and $v_m = p_m v^*_m$ (recall the definition of p_m in (2.3)). For any $m \in G(c^*, \xi, \delta^*)$ we define the linear functional π_m on $L^{\mathrm{sym}}_\infty(\mathbb{R})$ as in (2.21) and we assume that v^*_m and v_m are normalized in such a way that $\pi_m(v_m) = 1$. In Theorem 2.3 of ref. 9 it has been proved that there are $c_\pm$ and c' all positive so that $c_- e^{-2\alpha\xi} \leqslant \lambda_m \leqslant c_+ e^{-2\alpha\xi}$ and

$$v_m(x) \leqslant c_+ e^{-\alpha' |\xi - x|}, \qquad \alpha' = \alpha - c' e^{-2\alpha\xi}, \tag{8.4}$$

$$|v_m(x) - \tilde{m}'_\xi(x)| \leqslant c_+ e^{-2\alpha\xi} e^{\alpha |\xi - x|} \xi^4 \qquad \text{for} \quad |\xi - x| \leqslant \xi/2. \tag{8.5}$$

where μ is defined in (2.8) and recall $\tilde{m}_\xi(x) = \sqrt{\mu}\, \bar{m}(\xi - x)$. Let us remark that by choosing ξ large enough the parameter α' can be as close to α as needed. An easy consequence of (8.4) and (8.5) is the existence of a constant $c_3 > 0$ such that:

$$|\pi_m(\psi)| \leqslant c_3 \, \|\psi\|_\infty \qquad \forall \psi \in L_\infty^{\text{sym}}(\mathbb{R}). \tag{8.6}$$

In Theorem 2.4 of ref. 9 it is proved that there exists $\xi_0 > 1$ such that for any $\xi > \xi_0$ and $m \in G(c^*, \xi, \delta^*)$ there exist constants $d_\pm > 0$ so that if ψ is such that $\pi_m(\psi) = 0$,

$$\|e^{L_m t} \psi\|_\infty \leqslant d_+ e^{-d_- t} \, \|\psi\|_\infty. \tag{8.7}$$

Moreover there is $c_4 > 0$ such that for all $\psi \in L_\infty^{\text{sym}}(\mathbb{R})$

$$\|L_m^{-1} \psi\|_\infty \leqslant c_4 \lambda_m^{-1} \, \|\psi\|_\infty, \qquad \|L_m^{-1} [\psi - \pi_m(\psi)]\|_\infty \leqslant c_4 \, \|\psi - \pi_m(\psi)]\|_\infty. \tag{8.8}$$

In the case $m = m_\xi^0$ stronger results hold. In order to avoid heavy notation we abbreviate:

$$\bar{L}_\xi = L_{m_\xi^0}, \qquad \bar{p}_\xi = p_{m_\xi^0}, \qquad \bar{\lambda}_\xi = \lambda_{m_\xi^0}, \qquad \bar{v}_\xi^* = v_{m_\xi^0}^*, \qquad \bar{v}_\xi = v_{m_\xi^0}, \qquad \bar{\pi}_\xi = \pi_{m_\xi^0}.$$

In Theorem 11.1 of ref. 9 it is proved that there are $\eta \in (\alpha, 3\alpha/2)$, $\xi_1 > \xi_0$, and $c_5 > 0$ such that, for all $\xi > \xi_1$,

$$|\partial_\xi \bar{v}_\xi(x) - \tilde{m}''_\xi(x)| \leqslant c_5 e^{-\alpha\xi} \xi^4 \qquad \text{for} \quad |\xi - x| \leqslant \xi/2, \tag{8.9}$$

$$|\partial_\xi \bar{v}_\xi(x)| \leqslant c_5 \xi^2 \bar{v}_\xi(x) \qquad \text{for} \quad |\xi - x| \geqslant \xi/2, \tag{8.10}$$

$$\left| \frac{d\bar{\lambda}_\xi}{d\xi} \right| \leqslant c_5 e^{-\eta\xi}. \tag{8.11}$$

Setting

$$\partial_\xi \bar{\pi}_\xi(\psi) = \int_0^\infty dx \, \partial_\xi \bar{v}_\xi^*(x) \, \psi(x), \qquad \psi \in L_\infty^{\text{sym}}(\mathbb{R}), \tag{8.12}$$

from (8.9) and (8.10) it follows that, for some $c_6 > 0$,

$$|\partial_\xi \bar{\pi}_\xi(\psi)| \leqslant c_6 \, \|\psi\|_\infty. \tag{8.13}$$

Furthermore from Proposition 6.3 of ref. 9 (see also Eq. (6.38) of ref. 10), for any $\zeta \in (0, \alpha)$ there exists $c_7 > 0$ so that, for any $t, x, y > 0$,

$$e^{\bar{L}_\xi t}(x, y) \leqslant c_7 \bar{\lambda}_\xi t e^{\bar{\lambda}_\xi t} e^{-\zeta |x-y|}. \tag{8.14}$$

We now come back to the construction of n_ξ. The definition (8.1) is motivated by the fact that $f(m_\xi^0)$ (see (2.1)) is small for h small and ξ large. With this in mind and following ref. 10 we define:

$$b_\xi(x) \doteq -m_\xi^0(x) + \tanh\{\beta(J * m_\xi^0)(x)\}, \tag{8.15}$$

so that $f(m_\xi^0) = b_\xi + h\bar{p}_\xi + O(h^2)$. In Lemma 5.1 of ref. 10 it is shown that given $\alpha_0 > \alpha$ as in (2.7) there is $c_8 > 0$ so that, for all $\xi > 1$ and $x \geqslant 0$,

$$|b_\xi(x) + e^{-2\alpha\xi} k_\xi^0(x)| \leqslant c_8 (e^{-\alpha_0 \xi} \mathbf{1}_{0 \leqslant x \leqslant 1} + e^{-2\alpha(\xi + x)}), \tag{8.16}$$

where $k_\xi^0(x) = k^0(\xi - x)$ and

$$k^0(y) \doteq \frac{a e^{\alpha y}}{1 - m_\beta^2} [m_\beta^2 - \bar{m}^2(y)]. \tag{8.17}$$

Observe that from (2.7) it follows that there is $c_9 > 0$ such that

$$\|k^0\|_\infty \leqslant c_9. \tag{8.18}$$

From the estimate (8.16) the leading term in the projection of $f(m_\xi^0)$ is $V(\xi)/\sqrt{\mu}$ (with $V(\xi)$ as in (2.16)): this is the content of the next lemma.

Lemma 8.1. There is $\ell_0 > 1$ such that for any $h < e^{-2\alpha\ell_0}$ the following holds. There exists $\delta_3 > 0$ such that, for any $\kappa > 0$ and any $\xi \in \Gamma_{\ell_0, \kappa}$,

$$|\sqrt{\mu}\, \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi) - V(\xi)| \leqslant C e^{-(2\alpha + \delta_3)\xi}. \tag{8.19}$$

Proof. We choose $\ell_0 > \xi_1$ so large that $\alpha_0 + \alpha' > 2\alpha$ (α' as in (8.4)). Then, from (8.4) and (8.16), it is easy to check we can find $\delta_4 > 0$ such that:

$$|\bar{\pi}_\xi(b_\xi) + e^{-2\alpha\xi} \bar{\pi}_\xi(k_\xi^0)| \leqslant C e^{-(2\alpha + \delta_4)\xi}. \tag{8.20}$$

From (8.4) and (8.18) we have:

$$\bar{\pi}_\xi(k_\xi^0) = \int_{\xi/2}^{3\xi/2} dx \, \frac{\bar{v}_\xi(x)}{\bar{p}_\xi(x)} k_\xi^0(x) + O(e^{-\alpha'\xi/2}).$$

On the other hand, by the definitions (2.10), (8.17), recalling that $p_{\bar{m}} = \beta(1-\bar{m}^2)$, and using the estimates (2.7) and (8.18), we also have:

$$\sqrt{\mu}\, K = \sqrt{\mu} \int dx\, \frac{\bar{m}'(x)}{p_{\bar{m}}(x)}\, k^0(x) = \int_{\xi/2}^{3\xi/2} dx\, \frac{\tilde{m}'_\xi(x)}{p_{\tilde{m}_\xi}(x)}\, k^0_\xi(x) + O(e^{-\alpha\xi/2}).$$

From (8.1) and (8.5) we conclude that

$$|\bar{\pi}_\xi(k^0_\xi) - \sqrt{\mu}\, K| \leqslant Ce^{-\alpha'\xi/2}. \tag{8.21}$$

We finally observe that

$$\begin{aligned}\bar{\pi}_\xi(\bar{p}_\xi) &= 2m_\beta \sqrt{\mu} - \int_\xi^\infty dx\, \tilde{m}'(x) + \int_0^\infty dx (\bar{v}_\xi - \tilde{m}'_\xi)(x) \\ &= 2m_\beta \sqrt{\mu} + O(e^{-\alpha'\xi/2}),\end{aligned} \tag{8.22}$$

where in the last equality we used again (2.7), (8.4), and (8.5). From (8.20), (8.21), and (8.22) we get (8.19) for a suitable $\delta_3 > 0$. ∎

We now construct the function n_ξ in such a way that the main contribution to $f(n_\xi)$ is given by $\bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \bar{v}_\xi$.

Definition 8.2. Given $\xi > \ell_0$ we define the symmetric function ψ_ξ as the solution of

$$\bar{L}_\xi \psi_\xi + b_\xi + h\bar{p}_\xi - \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \bar{v}_\xi = 0. \tag{8.23}$$

The quasi-invariant manifold $\mathscr{M}_{\ell,\kappa}$ is then defined via the symmetric functions:

$$n_\xi(x) \doteq m^0_\xi(x) + \psi_\xi(x), \qquad \xi \in \Gamma_{\ell_0, +\infty}. \tag{8.24}$$

We note that the function ψ_ξ is well defined:

$$\psi_\xi = -\int_0^{+\infty} dt\, e^{\bar{L}_\xi t} \bar{T}_\xi (b_\xi + h\bar{p}_\xi),$$

where $\bar{T}_\xi$ is the projection acting on $L^{\text{sym}}_\infty(\mathbb{R})$, i.e., $\bar{T}_\xi \psi = \psi - \bar{\pi}_\xi(\psi)\, \bar{v}_\xi$. Observe that, by (8.4) and (8.6),

$$\|\bar{T}_\xi \psi\|_\infty \leqslant C\, \|\psi\|_\infty. \tag{8.25}$$

Lemma 8.3. In the same hypothesis of Lemma 8.1, there exists $\delta_5 > 0$ such that, for any $\kappa > 0$ and any $\xi \in \Gamma_{\ell_0,\kappa}$,

$$\|\psi_\xi\|_\infty \leqslant C e^{-(\alpha+\delta_5)\,\xi}, \qquad \|\partial_\xi \psi_\xi\|_\infty \leqslant C e^{-(\alpha+\delta_5)\,\xi}. \tag{8.26}$$

Proof. From (8.16), (8.18), and (2.12), since $\bar{p}_\xi \leqslant \beta$,

$$\|b_\xi + h\bar{p}_\xi\|_\infty \leqslant C(e^{-2\alpha\xi} + e^{-\alpha_0\xi}), \tag{8.27}$$

so that, by (8.8) and (8.25),

$$\|\psi_\xi\|_\infty \leqslant C(e^{-2\alpha\xi} + e^{-\alpha_0\xi}). \tag{8.28}$$

To compute the derivative of ψ_ξ we differentiate (8.23) and, recalling (8.12), we get:

$$\begin{aligned} \bar{L}_\xi\, \partial_\xi \psi_\xi + \partial_\xi \bar{p}_\xi J * \psi_\xi &= -\partial_\xi (b_\xi + h\bar{p}_\xi) + \bar{\pi}_\xi(\partial_\xi(b_\xi + h\bar{p}_\xi))\, \bar{v}_\xi \\ &\quad + \partial_\xi \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \bar{v}_\xi + \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \partial_\xi \bar{v}_\xi. \end{aligned} \tag{8.29}$$

We now proceed as in the proof of Proposition 4.2. Recalling the definition of the operator $\bar{T}_\xi$, we rewrite (8.29) as follows:

$$\begin{aligned} \bar{L}_\xi\, \partial_\xi \psi_\xi &= -\bar{T}_\xi(\partial_\xi \bar{p}_\xi J * \psi_\xi) - \bar{T}_\xi(\partial_\xi(b_\xi + h\bar{p}_\xi)) + \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \bar{T}_\xi\, \partial_\xi \bar{v}_\xi \\ &\quad - [\bar{\pi}_\xi(\partial_\xi \bar{p}_\xi J * \psi_\xi) - \partial_\xi \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi) - \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \bar{\pi}_\xi(\partial_\xi \bar{v}_\xi)]\, \bar{v}_\xi. \end{aligned}$$

Since $b_\xi + h\bar{p}_\xi = -\bar{L}_\xi \psi_\xi + \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \bar{v}_\xi$ then

$$\partial_\xi \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi) = -\partial_\xi \bar{\pi}_\xi(\bar{L}_\xi \psi_\xi) + \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \partial_\xi \bar{\pi}_\xi(\bar{v}_\xi). \tag{8.30}$$

We next observe that $\bar{\pi}_\xi(\partial_\xi \bar{v}_\xi) + \partial_\xi \bar{\pi}_\xi(\bar{v}_\xi) = 0$. We then get:

$$\begin{aligned} \bar{L}_\xi\, \partial_\xi \psi_\xi &= -\bar{T}_\xi[\partial_\xi(b_\xi + h\bar{p}_\xi) + \partial_\xi \bar{p}_\xi J * \psi_\xi - \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \partial_\xi \bar{v}_\xi] \\ &\quad - [\bar{\pi}_\xi(\partial_\xi \bar{p}_\xi J * \psi_\xi) + \partial_\xi \bar{\pi}_\xi(\bar{L}_\xi \psi_\xi)]\, \bar{v}_\xi. \end{aligned} \tag{8.31}$$

By standard arguments the derivative $\partial_\xi \psi_\xi$ is defined by the solution to (8.31), whose existence is guaranteed from the existence of $\bar{L}_\xi^{-1}$, see (8.8).

In order to prove the second inequality in (8.26) we need to bound the various terms on the right hand side of (8.31). We start with the term $\partial_\xi b_\xi$. Let R_ξ be the function defined as:

$$\begin{aligned} R_\xi(x) &= \{m_\xi^0(x) - [\bar{m}(\xi - x) - a e^{-\alpha(\xi + x)}]\}\, \mathbf{1}_{x \in [-1,0]} \\ &= \{[\bar{m}(\xi + x) + a e^{-\alpha(\xi + x)}] - [\bar{m}(\xi - x) + a e^{-\alpha(\xi - x)}]\}\, \mathbf{1}_{x \in [-1,0]}. \end{aligned}$$

Furthermore:

$$\partial_\xi R_\xi(x) = \{\bar{m}'(\xi+x) - a\alpha e^{-\alpha(\xi+x)} - \bar{m}'(\xi-x) + a\alpha e^{-\alpha(\xi-x)}\} \mathbf{1}_{x\in[-1,0]}. \tag{8.32}$$

Then, by (2.7),

$$\|R_\xi\|_\infty + \|\partial_\xi R_\xi\|_\infty \leqslant Ce^{-\alpha_0\xi}. \tag{8.33}$$

Since $m^0_\xi(x) = \bar{m}(\xi - x) - ae^{-\alpha(x+\xi)} + R_\xi(x)$ if $x > -1$, by Taylor expansion and using that $L_{\bar{m}}\bar{m}' = 0$, after some simple computations we easily get:

$$\begin{aligned}\partial_\xi b_\xi(x) = &-a\alpha L_{\bar{m}} e^{-\alpha(\cdot+\xi)}(x) + L_{\bar{m}}\, \partial_\xi R_\xi(x) \\ &+ \beta^2 \tanh''(\zeta)[-ae^{-\alpha(x+\xi)} + R_\xi(x)](J * \partial_\xi m^0_\xi)(x)\end{aligned}$$

where $\zeta = \zeta(x)$ is a number in the interval with end-points $\beta(J * \bar{m})(\xi - x)$ and $\beta(J * \bar{m})(\xi - x) - a(J * e^{-\alpha(\cdot+\xi)})(x) + (J * R_\xi)(x)$. By using (2.6) and (2.7), we have, for all $x \geqslant 0$,

$$\begin{aligned}|L_{\bar{m}} e^{-\alpha(\cdot+\xi)}(x)| &= e^{-\alpha(\xi+x)} \left|1 - \frac{(p_{\bar{m}} J * \bar{m})(\xi - x)}{1 - m_\beta^2}\right| \\ &\leqslant Ce^{-\alpha(\xi+x)} e^{-\alpha|\xi-x|} \leqslant Ce^{-3\alpha\xi/2}.\end{aligned} \tag{8.34}$$

By (8.33) $\|L_{\bar{m}}\, \partial_\xi R_\xi\|_\infty \leqslant Ce^{-\alpha_0\xi}$. Then, since $\|\tanh''\|_\infty \leqslant 2$, for $x \geqslant 0$,

$$\begin{aligned}&|\beta^2 \tanh''(\zeta)[-ae^{-\alpha(x+\xi)} + R_\xi(x)](J * \partial_\xi m^0_\xi)(x)| \\ &\quad\leqslant C(e^{-\alpha(\xi+x)} + e^{-\alpha_0\xi})\, e^{-\alpha|\xi-x|} \leqslant C(e^{-3\alpha\xi/2} + e^{-\alpha_0\xi}).\end{aligned}$$

By (8.34) and the above estimates, $\|\partial_\xi b_\xi(x)\|_\infty \leqslant C(e^{-3\alpha\xi/2} + e^{-\alpha_0\xi})$. From (2.12) and the fact that $\partial_\xi \bar{p}_\xi$ is uniformly bounded, we conclude that:

$$\|\partial_\xi(b_\xi + h\bar{p}_\xi)\|_\infty \leqslant C(e^{-3\alpha\xi/2} + e^{-\alpha_0\xi}). \tag{8.35}$$

To bound the second term on the right hand side of (8.31) we use again that $\partial_\xi \bar{p}_\xi$ is uniformly bounded and the bound (8.28), getting:

$$\|\partial_\xi \bar{p}_\xi J * \psi_\xi\|_\infty \leqslant C(e^{-2\alpha\xi} + e^{-\alpha_0\xi}). \tag{8.36}$$

The third term on the right hand side of (8.31) is bounded by using (8.19), (2.12) and (8.9), (8.10), obtaining

$$\|\bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\, \bar{T}_\xi(\partial_\xi \bar{v}_\xi)\|_\infty \leqslant Ce^{-2\alpha\xi}. \tag{8.37}$$

Finally, for the last terms on the right hand side of (8.31) we apply the same argument as in Eqs. (4.11) and (4.12) to which we refer for details. Then:

$$\bar{\pi}_\xi(\partial_\xi \bar{p}_\xi J * \psi_\xi) + \partial_\xi \bar{\pi}_\xi(\bar{L}_\xi \psi_\xi) = \bar{\lambda}_\xi \int_0^\infty dx\, \psi_\xi \left(\frac{\partial_\xi \bar{v}_\xi}{\bar{p}_\xi} - \frac{\partial_\xi \bar{p}_\xi}{\bar{p}_\xi^2}\, \bar{v}_\xi \right). \tag{8.38}$$

Therefore, by (8.28),

$$\|\bar{\pi}_\xi(\partial_\xi \bar{p}_\xi J * \psi_\xi) + \partial_\xi \pi_\xi(\bar{L}_\xi \psi_\xi)\|_\infty \leqslant C \bar{\lambda}_\xi (e^{-2\alpha\xi} + e^{-\alpha_0 \xi}). \tag{8.39}$$

From (8.8), (8.31), and the estimates (8.35), (8.36), (8.37), and (8.39) we finally obtain:

$$\|\partial_\xi \psi_\xi\|_\infty \leqslant C(e^{-3\alpha\xi/2} + e^{-\alpha_0 \xi}). \tag{8.40}$$

Setting $\delta_5 = \min\{\alpha/2;\, \alpha_0 - \alpha\}$, the inequalities (8.26) follow from (8.28) and (8.40). ▮

Proof of Theorem 2.2. Equations (2.13) and (2.14) follow from the definition (8.24) of n_ξ and the estimates (8.26). Recalling the definition (8.15), we next write:

$$f(n_\xi) = \tanh\{\beta[J * (m_\xi^0 + \psi_\xi) + h]\} - \tanh\{\beta J * m_\xi^0\} + b_\xi - \psi_\xi,$$

from which, by Taylor expansion to second order and using (8.23),

$$f(n_\xi) = \bar{L}_\xi \psi_\xi + h \bar{p}_\xi + b_\xi + R(\psi_\xi, h) = \bar{\pi}_\xi(b_\xi + h \bar{p}_\xi)\, \bar{v}_\xi + R(\psi_\xi, h),$$

where $|R(\psi_\xi, h)| \leqslant C(\|\psi_\xi\|_\infty^2 + h^2)$. From (8.19), (8.26), and (2.12) we get (for $\xi \in \Gamma_{\ell_0, \kappa}$):

$$\begin{aligned} &\|f(n_\xi) - V(\xi)\, \partial_\xi n_\xi\|_\infty \\ &\quad \leqslant C e^{-2\alpha\xi}\, \|\bar{v}_\xi - \sqrt{\mu}\, \partial_\xi n_\xi\|_\infty + C(e^{-(2\alpha+\delta_3)\,\xi} + e^{-2(\alpha+\delta_5)\,\xi} + e^{-4\alpha\xi}). \end{aligned}$$

On the other hand, by (8.4), (8.5), and the second inequality in (8.26),

$$\|\bar{v}_\xi - \sqrt{\mu}\, \partial_\xi n_\xi\|_\infty \leqslant C(e^{3\alpha\xi/2}\xi^4 + e^{-\alpha'\xi} + e^{-(\alpha+\delta_5)\,\xi}).$$

By the above bounds the estimate (2.15) follows for a suitable $\delta_0 > 0$. ▮

To prove Proposition 2.3 we have to show that $n_\xi + h \in G(c^*, \xi, \delta^*)$. To this end we need more refined estimates on ψ_ξ, and this is done in the next lemma. We first introduce some notation. We define

$$\Gamma_\xi \doteq \int_0^{\xi^2} dt\, e^{\bar{L}_\xi t} \bar{T}_\xi (b_\xi + h\bar{p}_\xi).$$

Then, from (8.7), (8.25), and (8.27),

$$\|\psi_\xi - \Gamma_\xi\|_\infty \leqslant C e^{-d_- \xi^2} (e^{-2\alpha\xi} + e^{-\alpha_0 \xi}). \tag{8.41}$$

Recalling (8.16) and (8.17), we write:

$$\begin{aligned} \Gamma_\xi &= P_\xi + Q_\xi, & P_\xi &\doteq \int_0^{\xi^2} dt\, e^{\bar{L}_\xi t} \bar{T}_\xi (b_\xi + k_\xi), \\ k_\xi(x) &\doteq e^{-2\alpha\xi} k_\xi^0(x), & Q_\xi &\doteq -\int_0^{\xi^2} dt\, e^{\bar{L}_\xi t} \bar{T}_\xi (k_\xi - h\bar{p}_\xi). \end{aligned} \tag{8.42}$$

Lemma 8.4. For each $\alpha^* < \alpha$ there exists $C_* > 0$ so that

$$|P_\xi(x)| \leqslant C_* [e^{-\alpha^*(\xi + x)} + e^{-2\alpha\xi} (e^{-\alpha^* \xi/4} \mathbf{1}_{|x-\xi| \leqslant \sqrt{\xi}} + \mathbf{1}_{|x-\xi| > \sqrt{\xi}})]. \tag{8.43}$$

Furthermore:

$$|Q_\xi(x)| \leqslant C e^{-2\alpha\xi}. \tag{8.44}$$

Proof. The proof of this lemma is similar to the proof of the second inequality in Equation (6.1) of ref. 10, therefore we only give an outline of the argument, by referring to ref. 10 for more details. We first prove (8.44). By (8.7), (8.18), (8.25), and (2.12) we have:

$$\|Q_\xi\|_\infty \leqslant C \int_0^{\xi^2} dt\, e^{-d_- t} (\|h\bar{p}_\xi\|_\infty + \|k_\xi\|_\infty) \leqslant C e^{-2\alpha\xi}. \tag{8.45}$$

Since $\bar{T}_\xi \psi = \psi - \bar{\pi}_\xi(\psi)\, \bar{v}_\xi$ and $\|\bar{v}_\xi\|_\infty \leqslant C$, by (8.16) and (8.20),

$$|P_\xi(x)| \leqslant C\xi^2 e^{-(2\alpha + \delta_4)\xi} + \int_0^{\xi^2} dt \int_0^{+\infty} dy\, e^{\bar{L}_\xi t}(x, y) (e^{-\alpha_0 \xi} \mathbf{1}_{y \in [0,1]} + e^{-2\alpha(\xi + y)}).$$

An estimate of the last integral can be found in ref. 10 but, since it is not written in the form we need, we give the argument below. From (8.4) and (8.14) we get, for any $x \geqslant 0$,

$$\int_0^{\xi^2} dt \int_0^{+\infty} dy\, e^{\bar{L}_\xi t}(x, y)\, e^{-\alpha_0 \xi} \mathbf{1}_{y \in [0,1]} \leqslant C\xi^2 e^{-(\alpha_0 \xi + \zeta x)}. \tag{8.46}$$

Again by (8.14),

$$\int_0^{\xi^2} dt \int_0^{+\infty} dy\, e^{\bar{L}_\xi t}(x, y)\, e^{-2\alpha(\xi + y)} \leqslant C\xi^2 e^{-2\alpha\xi} \int_0^{+\infty} dy\, e^{-\zeta\,|x-y|} e^{-2\alpha y}.$$

The integral on the r.h.s. is uniformly bounded; moreover, if $|\xi - x| \leqslant \sqrt{\xi}$ then, since $|x-y| \geqslant |y-\xi| - |x-\xi|$,

$$\begin{aligned}\int_0^{+\infty} dy\, e^{-\zeta\,|x-y|} e^{-2\alpha y} &= \int_0^{+\infty} dy\, e^{-\zeta\,|x-y|} e^{-2\alpha y} (\mathbf{1}_{|y-\xi| \leqslant \xi/2} + \mathbf{1}_{|y-\xi| > \xi/2}) \\ &\leqslant C(e^{-3\alpha/2} + e^{\zeta(\sqrt{\xi} - \xi/2)}).\end{aligned}$$

Collecting all the above estimates we obtain (8.43) by choosing $\alpha^* < \zeta$ (recall that in (8.14) the parameter $\zeta \in (0, \alpha)$ can be fixed arbitrarily close to α). ∎

Proof of Proposition 2.3. From (2.12), (8.41), (8.43), and (8.44) we easily get that $\psi_\xi + h$ satisfies the bounds on the right hand side of (8.2) for a suitable constant c^* if $\xi > \ell_0$ is chosen large enough. Instead the estimate (8.3) is not immediate. By (8.41), (8.42), and (8.43) there is $\delta_6 > 0$ so that

$$\left| \int_{|x-\xi| \leqslant \sqrt{\xi}} dx [\psi_\xi(x) + h - Q_\xi(x)]\, \bar{m}'(\xi - x)^2\, \bar{m}(\xi - x) \right| \leqslant C e^{-(2\alpha + \delta_6)\,\xi}. \tag{8.47}$$

(note that since $(\bar{m}')^2\, \bar{m}$ is antisymmetric the constant $+h$ does not contribute to the integral). The contribution of Q_ξ to the above integral has been estimated in Section 8 of ref. 10. More precisely, setting

$$M_2 \doteq \int_{|x-\xi| \leqslant \sqrt{\xi}} dx\, Q_\xi(x)\, \bar{m}'(\xi - x)^2\, \bar{m}(\xi - x),$$

it is proved that there is δ_7 such that $|M_2 - M_3| \leqslant C e^{-(2\alpha + \delta_7)\,\xi}$, where

$$M_3 \doteq -\int_{|x-\xi| \leqslant \sqrt{\xi}} dx\, \bar{m}'(\xi - x)^2\, \bar{m}(\xi - x) \int_0^{\xi^2} dt \int dy\, G_{\xi, t}(x, y)(h\bar{p}_\xi - k_\xi)(y),$$

with $G_{\xi,t}(x,y)$ explicitly given in Eq. (8.10) of ref. 10 and satisfying $G_{\xi,t}(\xi-x, y-\xi)=G_{\xi,t}(\xi+x, y+\xi)$. It is finally shown that M_3 is bounded by the r.h.s. of (8.3): here we only remark that, since $(\bar{m}')^2\,\bar{m}$ is antisymmetric, only the odd part (w.r.t. ξ) of $h\bar{p}_\xi - k_\xi$ contributes to the integral, and since $\bar{p}_\xi$ is even, only the odd part of k_ξ survives. The estimate of this last term uses the fact that J is nonincreasing, we refer to ref. 10 for the details.

We have thus proved that for any $\kappa>0$ there exists $\ell_1>\ell_0$ and suitable positive constants c^* and δ^* such that $n_\xi+h\in G(c^*,\xi,\delta^*)$ for all $\xi\in\Gamma_{\ell_1,\kappa}$. Therefore Proposition 2.3 as well as the estimates (2.23) and (2.22) follow from the spectral analysis in $G(c^*,\xi,\delta^*)$. ▮

Proof of Lemma 2.4. The proof of inequalities (2.24) and (2.25) can be done following line by line that one of (8.9), (8.10), and (8.11) given in Section 11 of ref. 9. The arguments given there apply also in our case (where m^0_ξ is replaced by n_ξ): the key tool is Lemma 8.3, which gives sufficiently strict bounds on $n_\xi - m^0_\xi$. Finally, recalling that $v_\xi = p_\xi v^*_\xi$ and that $\|\partial_\xi p_\xi\|_\infty \leqslant C$, (2.26) follows from (2.18) and (2.25). We next prove (2.27)–(2.29). By the second inequality in (8.26) and recalling (8.1) we have:

$$\|\sqrt{\mu}\,\partial_\xi n_\xi - \tilde{m}'_\xi\|_\infty \leqslant Ce^{-\alpha\xi}. \tag{8.48}$$

On the other hand, since $n_\xi\in G(c^*,\xi,\delta^*)$, from (8.4) and (8.5) it follows that $\|v_\xi - \tilde{m}'_\xi\|_\infty \leqslant Ce^{-\alpha'\xi/2}$, $(v_{n_\xi}=v_\xi)$. Then, since $\pi_\xi(v_\xi)=1$, from (2.22) we get $|\sqrt{\mu}\,\pi_\xi(\partial_\xi n_\xi)-1|\leqslant Ce^{-\alpha'\xi/2}$. Collecting together the above estimates, (2.27) and (2.28) follow. We are left with the proof of (2.29). By (2.18), (2.24), (2.25), (8.26), and (8.48) we have:

$$\left|\partial_\xi \pi_\xi(\partial_\xi n_\xi) - \sqrt{\mu}\int_0^{+\infty} dx\,\frac{\bar{m}''(\xi-x)\,\bar{m}'(\xi-x)}{p_{\bar{m}}(\xi-x)}\right| \leqslant Ce^{-\alpha'\xi/2}.$$

But, since $\bar{m}$ is odd and $p_{\bar{m}}$ even,

$$\int dx\,\frac{\bar{m}''(\xi-x)\,\bar{m}'(\xi-x)}{p_{\bar{m}}(\xi-x)} = 0 \qquad \forall \xi>0.$$

Then by (2.7) and the previous estimate (2.29) follows. ▮

Proof of Proposition 2.10. The critical droplet $q=n_{\bar{\xi}}+\bar{\varphi}$ belongs to the set $G(c^*,\bar{\xi},\delta^*)$ for h small. The existence and uniqueness of $\gamma_v>\gamma$ solving (2.45) follows from Lemma 3.1 of ref. 9. The proof of (2.46) is not given in ref. 9, but it can be done in the same way as the proof of (2.41) in Section 5. In fact, recalling the definition (2.42) of the linear operator L, we can rewrite the equation for the eigenfunction v as $v=(1+\lambda)^{-1}\, pJ*v$; then, by arguing as in Section 5, we have:

$$v(x+\xi_q)=\int_{s-1}^{s} dy\, \tilde{G}_s(x,y)\, v(y+\xi_q), \qquad \forall x>s\geqslant \ell,$$

with ξ_q, ℓ as in (5.19) and

$$\tilde{G}_s(x,y)\doteq \sum_{n=1}^{\infty} \frac{1}{(1+\lambda)^n}\int_s^{+\infty} dy_1\cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^{n} p_{\xi_q}(y_{i-1})\, J(y_{i-1}-y_i).$$

Finally (2.47) is exactly Eq. (2.23) of ref. 9, the only difference here is that this estimate can be proved also for $\zeta=\gamma_v$, as it can be checked by exploiting the proof in Section 10 of ref. 9. The others statements of Proposition 2.10 follow from the spectral analysis in the set $G(c^*, \bar{\xi}, \delta^*)$. ▮

Proof of Theorem 2.5. In Theorem 2.2, Proposition 2.3, and Lemma 2.4 we have introduced functions $\ell_i=\ell_i(\kappa)$ and $c_i=c_i(\kappa)$ for $i=0,1,2$ (which we can assume nondecreasing). In the sequel, given $\bar{\kappa}>0$, we shorthand $\bar{\ell}_i=\ell_i(\bar{\kappa})$ and $\bar{c}_i=c_i(\bar{\kappa})$, $i=0,1,2$. For $\varepsilon>0$ and $\bar{\ell}>\bar{\ell}_2$ to be fixed, we define:

$$F\colon \Gamma_{\bar{\ell},\bar{\kappa}}\times \mathscr{B}_{\bar{\ell},\bar{\kappa},\varepsilon}\to \mathbb{R}\colon F(\xi,m)=\pi_\xi(n_\xi-m).$$

Clearly $F(\xi,n_\xi)=0$ for all $\xi\in\Gamma_{\bar{\ell},\bar{\kappa}}$. Given $\ell_3>\bar{\ell}$ and $\kappa<\bar{\kappa}$, we next apply the contraction mapping principle in the first argument of the function

$$G\colon \Gamma_{\bar{\ell},\bar{\kappa}}\times \mathscr{B}_{\bar{\ell},\bar{\kappa},\varepsilon}\times \Gamma_{\ell_3,\kappa}\to \mathbb{R}\colon G(\xi,m,\xi^*)=\xi-[\partial_\xi F(\xi^*,n_{\xi^*})]^{-1}\, F(\xi,m).$$

Since $\partial_\xi F(\xi^*,n_{\xi^*})=\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})$ then

$$\partial_\xi G=\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})^{-1}\,[\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})-\pi_\xi(\partial_\xi n_\xi)+\partial_\xi \pi_\xi(n_\xi-m)],$$

from which, by using (2.26) and (2.28),

$$|\partial_\xi G|\leqslant (\sqrt{\mu}+\bar{c}_2)(|\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})-\pi_\xi(\partial_\xi n_\xi)|+\bar{c}_2\,\|m-n_\xi\|_\infty).$$

But (2.14) implies $\|n_\xi-n_{\xi^*}\|_\infty\leqslant(\bar{c}_0+\|\bar{m}'\|_\infty)\,|\xi-\xi^*|$, hence there is a constant $C=C(\bar{\kappa})$ so that:

$$|\partial_\xi G|\leqslant C(|\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})-\pi_\xi(\partial_\xi n_\xi)|+|\xi-\xi^*|+\|m-n_{\xi^*}\|_\infty).$$

Using again (2.28), we can now fix $\bar{\ell}$ so large that (for all $h<e^{-2\alpha\ell_3}$)

$$C\,|\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})-\pi_\xi(\partial_\xi n_\xi)|<\tfrac{1}{8}.$$

Then, if $|\xi-\xi^*|<r$ and $\|m-n_{\xi^*}\|_\infty<\bar{\varepsilon}$ with $r, \bar{\varepsilon}>0$ small enough, we find $|\partial_\xi G|<1/4$ (for all $h<e^{-2\alpha\ell_3}$). Moreover, if we assume also $\ell_3>\bar{\ell}+r$ and $\kappa<\bar{\kappa}-r$, the function $G(\cdot, m, \xi^*)$ is defined for all $\xi\in(\xi^*-r, \xi^*+r)$ and

$$\begin{aligned}|G(\xi, m, \xi^*)-\xi^*| &\leqslant |G(\xi, m, \xi^*)-G(\xi, n_{\xi^*}, \xi^*)| \\ &\quad +|G(\xi, n_{\xi^*}, \xi^*)-G(\xi^*, n_{\xi^*}, \xi^*)| \\ &\leqslant |\pi_{\xi^*}(\partial_{\xi^*}n_{\xi^*})|^{-1}\,|\pi_\xi(n_{\xi^*}-m)|+\frac{|\xi-\xi^*|}{4} \\ &\leqslant \bar{c}_2(\sqrt{\mu}+\bar{c}_2)\,\|n_{\xi^*}-m\|_\infty+\frac{|\xi-\xi^*|}{4} \\ &\leqslant \bar{c}_2(\sqrt{\mu}+\bar{c}_2)\,\varepsilon+\frac{r}{4}<\frac{r}{2},\end{aligned}$$

where we used (2.22), (2.28), and we further assumed $\bar{\varepsilon}$ so small that $4\bar{c}_2(\sqrt{\mu}+\bar{c}_2)\,\bar{\varepsilon}<r$. From the contraction mapping principle we get that there is a unique solution $\xi=\Xi(m, \xi^*)$ of the equation $F(\xi, m)=0$ in (ξ^*-r, ξ^*+r) for any $\xi^*\in\Gamma_{\ell_3,\kappa}$ and m such that $\|m-n_{\xi^*}\|_\infty<\bar{\varepsilon}$. Moreover $m\mapsto\Xi(m, \xi^*)$ is C^1 and $|\Xi(m, \xi^*)-\xi^*|<r/2$.

We finally observe that if ℓ_3 is large enough there is $\varepsilon_0\in(0, \bar{\varepsilon}]$ such that the following holds. If $\xi^*, \xi^{**}\in\Gamma_{\ell_3,\kappa}$ and $\|n_{\xi^*}-n_{\xi^{**}}\|_\infty<2\varepsilon_0$ then $|\xi^*-\xi^{**}|<r/2$. In fact, by using (2.7) and (2.14), from the condition $|\xi^*-\xi^{**}|\geqslant r/2$ it follows there is $C=C(\bar{\kappa})$ so that

$$|n_{\xi^*}(0)-n_{\xi^{**}}(0)|\geqslant(\alpha a e^{-\alpha\ell_3}-Ce^{-\min\{\alpha_0;\,\alpha+\delta_0\}\,\ell_3})\,\frac{r}{2},$$

and, for ℓ_3 large enough, the r.h.s. is not smaller than $2\varepsilon_0$ for any sufficiently small ε_0.

We now prove Theorem 2.5. Given $\kappa>0$, we fix $\bar{\kappa}>\kappa$ and then we introduce the parameters ε_0, $\bar{\ell}$, and ℓ_3 so that all the previous statements hold. Then we define the map $\Xi\colon\mathscr{B}_{\ell_3,\kappa,\varepsilon_0}\to\Gamma_{\bar{\ell},\bar{\kappa}}$ by setting $\Xi(m)=\Xi(m, \xi^*)$ if $\|m-n_{\xi^*}\|_\infty<\varepsilon_0$ (the existence of such $\xi^*\in\Gamma_{\ell_3,\kappa}$ follows from (2.30)). We see that the definition is well posed: assume $\|m-n_{\xi^{**}}\|_\infty<\varepsilon_0$ for some other $\xi^{**}\in\Gamma_{\ell_3,\kappa}$, hence $\|n_{\xi^*}-n_{\xi^{**}}\|_\infty<2\varepsilon_0$ and then $|\xi^*-\xi^{**}|\leqslant r/2$ for what stated before. Then $|\Xi(m, \xi^{**})-\xi^*|\leqslant|\Xi(m, \xi^{**})-\xi^{**}|+|\xi^{**}-\xi^*|<r$ and, by local uniqueness, $\Xi(m, \xi^{**})=\Xi(m, \xi^*)$. Moreover, to prove (2.31), we observe that $\|m-n_\xi\|_\infty\leqslant\|m-n_{\xi^*}\|_\infty+(\sqrt{\mu}+\bar{c}_2)\,|\xi-\xi^*|$ and, on the other hand, if $\xi=\Xi(m, \xi^*)$ then $|\xi-\xi^*|\leqslant\text{const}\,|\pi_{\xi^*}(\partial_{\xi^*}n_{\xi^*})|^{-1}\,|\pi_{\xi^*}(n_{\xi^*}-m)|\leqslant C\,\|m-n_{\xi^*}\|_\infty$, for some $C=C(\bar{\kappa})$. This last bound also prove (2.32), since $\|m-n_{\xi_0}\|_\infty<\varepsilon_0$ implies $\xi=\Xi(m, \xi_0)$.

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